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Étale Cohomology

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Exercise sheet 5[1]

Exercise 1. Show that a field extension is étale if only if it is separable and finite. (Hint: Hilbert's Nullstellensatz)

Exercise 2. Let $f : X \to Y$ be a map of schemes locally of finite presentation. Show that the subset $\{x \in X | x \text{ is unramified over } Y\}$ is an open subset of X. (Hint: Using what we have proved today, one can see that $x \in X$ is unramified over Y iff $\Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$=0.)

Remark 1. It is a non-trivial fact that the flat points of f is an open subset. Using this fact one can conclude that the subset of étale points of f is open in X.

Exercise 3. Let $f : X \to Y$, $g : Y \to Z$ be maps of schemes with g finite étale. Show that if $g \circ f$ is flat then f is flat. (Hint: Use Ex 4.4) Is it still true if g is not étale?

Exercise 4. Let $f, g : X \to Y$ be two morphisms between two S-schemes. In the class we have seen that if X is connected and Y/S is étale separated, then $f = g$ if and only if they agree on a geometric point of X. Find examples to see that the conclusion is false when Y/S is not étale.

Exercise 5. Let $f : X \to Y$ be a étale morphism between two schemes. In the class we have seen that if each geometric fiber $f^{-1}(y)$ consists of exactly one point and if f is finite, then f is an isomorphism. Show that the statement is still true removing the finiteness assumption. Is it still true when f is not étale, e.g. when f is only faithfully flat? Justify your conclusion with examples.

[1] If you want your solutions to be corrected, please hand them in just before the lecture on November 23, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.