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Étale Cohomology

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Exercise sheet 4[1]

Exercise 1. Let X be a locally Noetherian scheme. Show that the category of affine (resp. finite, finite flat) schemes over X is equivalent to the category of quasi-coherent (resp. coherent, vector bundles) $\mathcal{O}_X$-algebras over X.

Remark 1. The assumption *locally Noetherian* is not needed for affine morphisms. For finite and finite flat morphisms, if one remove the condition that X is locally Noetherian, then one should replace *coherent* by *modules of finite type* and *finite flat* by *finite locally free*.

Exercise 2. Let $f : X \to Y$ be a morphism of schemes. The map f is called a monomorphism if for two morphisms $g_1, g_2 : T \to X$, $f \circ g_1 = f \circ g_2$ if and only if $g_1 = g_2$.

(1) Show that if f is a monomorphism then for any $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ we have $X_1 \times_X X_2 = X_1 \times_Y X_2$.
(2) Let $y \in Y$ be a point. Show that the canonical map $\mathrm{Spec}\,(\mathcal{O}_{Y,y}) \to Y$ is a monomorphism of schemes.
(3) Let $y \in Y$ be a point. Show that the canonical map $\mathrm{Spec}\,(\kappa(y)) \to Y$ is a monomorphism of schemes.
(4) Show that open embeddings and closed embeddings are monomorphisms of schemes.

Exercise 3. We have seen that if we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

[1]If you want your solutions to be corrected, please hand them in just before the lecture on November 16, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.

in which g is faithfully flat and quasi-compact or faithfully flat and locally of finite presentation then f' is injective implies that f is injective. Is the converse true, i.e. if f is injective does it mean that f' is also injective?

Exercise 4. Let $f : X \to Y$ be a finite étale surjective morphism of schemes of degree n, where the degree of f is defined to be the rank of the locally free $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ (which is equal to the number of points of the geometric fibers $f^{-1}(\bar{y})$ (see the notation in the lecture notes in Prop. 3.2) for all $y \in Y$). Show that there is a finite étale surjective morphism $Y' \to Y$ such that $Y' \times_Y X$ is the n-copies of Y'. (Hint: Use Ex 3.3 and induction.)