October 27, 2016

# Étale Cohomology

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## Exercise sheet 2[1]

**Exercise 1.** Let $f : X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point. The residue field $\kappa(y)$ of $y$ is the quotient of $\mathcal{O}_{Y,y}$ by its maximal ideal. We have a canonical map of schemes $\mathrm{Spec}\,(\kappa(y)) \to Y$. Show that the fiber $f^{-1}(y)$ of $y$, which is defined to be the fibred product $X \times_Y \mathrm{Spec}\,(\kappa(y))$, is a scheme whose underlying topological space is the inverse image of $y$ under $f$ as a map of topological spaces. Using Ex 1.5 to show that $f$ is quasi-finite if and only if $f^{-1}(y)$ is a finite $\kappa(y)$-scheme for all $y$ and $f$ is of finite type.

**Exercise 2.** Let $f : X \to Y$ be a morphism of schemes. Show the following:

(1) The map $f$ is separated (resp. proper) if and only if there exists an open covering $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i) \to V_i$ is separated (resp. proper).
(2) The composite of two separated (resp. proper) morphisms is separated (resp. proper).
(3) Let $f : X \to Y$ and $\alpha : Y' \to Y$ be morphisms. Let $f' : X \times_Y Y' \to Y'$ be the base change map. Then $f$ is separated (resp. proper) implies that $f'$ is separated (resp. proper).
(4) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If $g \circ f$ is separated (resp. proper) and $g$ is any map (resp. separated), then $f$ is separated (resp. proper).

**Exercise 3.** Let $f : X \to Y$ be a locally of finite type morphism between locally Noetherian schemes. Show the following:

(1) Recall that a subset $Z \subseteq X$ is called constructible if $Z$ can be written as a finite disjoint union of locally closed subsets, i.e. the intersection of a closed subset and an open subset. A theorem of Chevalley assures that $f$ sends constructible subsets of $X$ to those of $Y$. Show that a constructible set $U \subseteq Y$ is

[1] If you want your solutions to be corrected, please hand them in just before the lecture on November 3, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.

open if it satisfies the following: If $u \in U$ and if for any point $v \in Y$ with the property that $u$ is contained in the closure of $v$ then $v \in U$.

(2) Let $A$ and $B$ be local rings with maximal ideal $\mathfrak{p}$ and $\mathfrak{q}$ respectively, and let $\phi : A \to B$ is a flat map of rings such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Show that for any $A$-module $M \neq 0$ the $B$-module $M \otimes_A B \neq 0$. (Hint: Take a non-zero element $m \in M$ then reduce to the case when $M = A/I$.)

(3) The assumptions are as in (2). Let $\mathfrak{p}' \subseteq \mathfrak{p}$ be another prime ideal of $A$ and let $S := A \setminus \mathfrak{p}'$. Show that $(S^{-1}\mathfrak{p}')(S^{-1}B) \neq (S^{-1}B)$. (Hint: Note that $\kappa(\mathfrak{p}') \otimes_A B = (S^{-1}B)/(S^{-1}\mathfrak{p}')(S^{-1}B)$, so we can apply (2).)

(4) The assumptions being as in (2), show that there is a prime $\mathfrak{q}' \subseteq B$ such that $f^{-1}(\mathfrak{q}') = \mathfrak{p}'$.

(5) Show that if $f$ is flat then it is open.