November 24th, 2015

Number Theory I

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Exercise sheet 9[1]

Exercise 1. Let A be a ring and let $A[[X]]$ be the ring of formal power series with coefficients in A. Prove that if A is Noetherian then $A[[X]]$ is Noetherian.

Exercise 2. Let A be a ring and let $A[X]$ be the ring of polynomials with coefficients in A. Suppose that $A[X]$ is a Noetherian ring, is A necessarily Noetherian?

Exercise 3. Let A be a ring such that every local ring A_P for P prime ideal of A is Noetherian. Is A necessarily Noetherian?

In the following k is an *algebraically closed field*. Let $A = k[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in k and $I \subset A$ an ideal. We denote by

$$Z(I) = \{a \in k^n \mid f(a) = 0, \text{ for all } f \in I\}$$

the zero set of I. Notice if $I = (f_1, \dots, f_r)$, then $Z(I) = \{a \in k^n \mid f_i(a) = 0, \text{ for all } i = 1, \dots, r\}$. Further we set

$$I(Z(I)) = \{f \in A \mid f(a) = 0 \text{ in } k, \text{ for all } a \in Z(I)\}.$$

Notice that $I(Z(I))$ is an ideal in A, which clearly contains $\sqrt{I}$ the radical of I.

Exercise 4. Let k be an algebraically closed field and $I \subset A := k[x_1, \dots, x_n]$ an ideal. Show with the notations from above:

(i) We have a bijection

$$Z(I) \xrightarrow{\simeq} \{\text{maximal ideals in } A/I\}, \quad (a_1, \dots, a_n) \mapsto (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n).$$

In particular, $Z(I) \neq \emptyset \Leftrightarrow I \neq (1)$. (*Hint*: Use the weak Hilbert Nullstellensatz to prove the surjectivity.)

(ii)

$$I(Z(I)) = \sqrt{I}.$$

(*Hint*: Take $f \in I(Z(I))$. To show that $f^N \in I$ for some $N \geq 1$ proceed as follows: Denote by A_f the ring of fractions of A with respect to the multiplicatively closed subset $S^{-1} :=$

[1]If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on December 18th.

$\{1, f, f^2, f^3, \ldots\}$. Show that $A_f \cong A[y]/(1 - fy)$ and hence is a finitely generated k-algebra. Show that $IA_f = A_f$ using the weak Hilbert Nullstellensatz and (i). Now conclude.)