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Number Theory I

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Exercise sheet 8[1]

Exercise 1. Let $k[x, y, z]$ be the polynomial ring in three variables over a field k. Let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (y, z)$, $\mathfrak{m} = (x, y, z)$, $\mathfrak{a} := \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$. What are the associated primes of $\mathfrak{a}$? Which of them are minimal and which of them are embedded?

Exercise 2. Let A be a commutative ring with no non-zero nilpotent element. Show that if the 0 ideal has a primary decomposition, then all the prime ideals associated to 0 are minimal prime ideals.

Exercise 3. Let A be a commutative ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the inverse image of $S^{-1}\mathfrak{a}$ under the localization map $A \to S^{-1}A$. Prove that

(i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$.

(ii) $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$, where $r(\mathfrak{a})$ means the radical ideal of $\mathfrak{a}$.

(iii) $S(\mathfrak{a}) = A \Leftrightarrow \mathfrak{a}$ meets S.

(iv) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$, where S_1, S_2 are two multiplicatively closed subsets of A.

(v) If $\mathfrak{a}$ has a primary decomposition, then the set of ideals $S(\mathfrak{a})$, when S runs through all multiplicatively closed subsets of A, is a finite set.

Exercise 4. Let A be a commutative ring, $\mathfrak{p}$ be a prime ideal. In the notation of the last exercise (Ex 8.3), we denote $\mathfrak{p}^{(n)} := S_{\mathfrak{p}}(\mathfrak{p}^n)$ where $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$. Prove that

(i) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;

(ii) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary component;

(iii) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is a $\mathfrak{p}$-primary component;

(iv) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \Leftrightarrow \mathfrak{p}^n$ is $\mathfrak{p}$-primary.

[1] If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on December 11th.