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Number Theory I

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Exercise sheet 7[1]

Exercise 1. Let A and B be rings, and let $f : A \to B$ be a homomorphism of rings; we consider B as an A-module with the structure induced by f. Let M and N be two A-modules.

(i) Show that there is a map of B-modules

$$\sigma : \mathrm{Hom}_A(M,N) \otimes_A B \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$$

(ii) Let B be a flat A-module, and let M be a finitely generated A-module; show that σ is injective.

(iii) Deduce from (i) and (ii) that if S is a multiplicatively closed subset of A, then there is a homomorphism of $S^{-1}(A)$-modules

$$\tau : S^{-1}(\mathrm{Hom}_A(M,N)) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

and prove that τ is injective if M is a finitely generated A-module.

(iv) Consider τ for $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus 0$, $N = M = \oplus_{n \geq 2}(\mathbb{Z}/n\mathbb{Z})$, show that in this case τ is not injective.

Exercise 2. Let A be a ring and write $\mathrm{Spec}\, A$ for the set of prime ideals of A. For any ideal $I \subset A$ write $V(I)$ for the set of prime ideals $P \in \mathrm{Spec}\, A$ containing I. If $\phi : A \to B$ is a homomorphism of rings, we know that $P \mapsto \phi^{-1}(P)$ induces a map $\phi^* : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

(i) If $S \subset A$ is a multiplicatively closed subset, and $\phi : A \to S^{-1}A$ is the localization morphism, show that ϕ^* is injective.

(ii) If $f \in A$, write $S_f := \{f^n, n \geq 0\}$, where $f^0 := 1$. This is a multiplicatively closed subset of A; again write $\phi : A \to S_f^{-1}A$ for the localization map. Show that $\phi^* : \mathrm{Spec}\, S_f^{-1}A \to \mathrm{Spec}\, A$ is injective and that its image is $(\mathrm{Spec}\, A) \setminus V((f))$. What happens if f is nilpotent?

Exercise 3. Let A be an integral domain and let M be an A-module. An element $m \in M$ is called *torsion element, or A-torsion element* if there exists $a \in A \setminus \{0\}$ such that $am = 0$. Write $T_A(M)$ for the set of A-torsion elements. Prove that:

(i) $T_A(M)$ is an A-submodule of M.

[1] If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on December 4th.

(ii) For any multiplicatively closed set $S \subset A$, there is an isomorphism of $S^{-1}A$-modules $S^{-1}T_A(M) \xrightarrow{\cong} T_{S^{-1}A}(S^{-1}M)$.

(iii) The following statements are equivalent:

(a) $T_A(M) = 0$.

(b) $T_{A_{\underline{P}}}(M_{\underline{P}}) = 0$ for all prime ideals $\underline{P} \subset A$.

(c) $T_{A_{\underline{M}}}(M_{\underline{M}}) = 0$ for all maximal ideals $\underline{M} \subset A$.

(iv) Now assume that A contains zero-divisors. Is $T_A(M) \subset M$ still an A-submodule? If not, find a counterexample.

Exercise 4. Let A be a ring which is product of two integral domains A_1, A_2, *i.e.* $A = A_1 \times A_2$. Find a minimal primary decomposition of 0.

Exercise 5.

(i) Let A and B be rings, and let $f : A \to B$ be an homomorphism of rings. If Q is a P-primary ideal of B, then show that $f^{-1}(Q)$ is a $f^{-1}(P)$-primary ideal of A.

(ii) Using (i) show that the ideal $(Y^2, X - YZ)$ is a (X, Y)-primary ideal in $K[X, Y, Z]$, where K is a field.