November 10th, 2015

Number Theory I

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Exercise sheet 5[1]

Exercise 1. If A is ring, M an A-module, and $I \subseteq A$ an ideal, then prove that there is an isomorphism of A-modules

$$(A/I) \otimes_A M \cong M/IM.$$

Exercise 2. Let A be a ring.

(i) Let M, N be two finitely generated A-modules, show that $M \otimes_A N$ is finitely generated.

(ii) Recall the definition of the *rank* of a finitely generated free A-module from the exercise sheet number 3. If M, N are free A-modules of finite rank, show that $M \otimes_A N$ is a free A-module of finite rank. Compute the rank of $M \otimes_A N$ in terms of the rank of M and the rank of N.

(iii) Suppose that A is a local ring and let M, N be finitely generated A-modules. If $M \otimes_A N = 0$, then either $M = 0$ or $N = 0$. (*Hint*: Let m be the maximal ideal of A, and let k be A/m, consider $M_k = k \otimes_A M$, which is isomorphic to M/mM by exercise 1. Then use Nakayama's lemma to prove that $M_k = 0$ implies $M = 0$. Then prove that $M \otimes_A N = 0$ implies that $M_k = 0$ or $N_k = 0$.)

Exercise 3. Let A, B be rings and let $f : A \to B$ a ring homomorphism.

(i) If M, N are two flat A-modules, then so is $M \otimes_A N$.

(ii) If A is a field then any A-module is flat.

(iii) We consider B as A-module via restriction of scalars given by f. If B is a flat A-module and N is a flat B-module, then N is flat as an A-module.

(iv) If M is a flat A-module, and B is considered as A-module via the map f, then $M \otimes_A B$ is a flat B-module.

Exercise 4. Let A be a ring and let M be an A-module.

(i) Let $a \neq 0 \in A$ which is not a zero divisor, and suppose that M is a flat A-module, then if $am = 0$ then $m = 0$ where $m \in M$.

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on November 20th.

(ii) If $I \subseteq A$ is an ideal, M is a flat A-module, then the natural map $I \otimes_A M \to IM$ sending $i \otimes m \mapsto im$ is an isomorphism.

Exercise 5. Let K be a field, and let $A := K[X, Y]$ be the polynomial ring in two variables with coefficients in K. Which of the following A-modules is flat?

(i) $K[X, Y, Z]$
(ii) $K[X, Y]/(X, Y)$
(iii) $K[X, Y]/(Y^2)$
(iv) $K[X, Y, Z]/(Z^2)$
(v) $K[X, Y, Z]/(XZ, YZ, Z^2)$