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Number Theory I

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Exercise sheet 3[1]

Exercise 1. Let A be a ring and let I be an ideal of A, and let Λ be a non empty set, and let M_λ be an A-module for each $\lambda \in \Lambda$. Prove that if $M_\lambda \subseteq M$ with M an A-module $I(\sum_\lambda M_\lambda) = \sum_\lambda (IM_\lambda)$. Prove that if I is finitely generated then $I(\prod_\lambda M_\lambda) = \prod_\lambda (IM_\lambda)$

Exercise 2. Let A be a ring and let M be an A-module, in class we defined the ideal $\mathrm{Ann}(M) = \{a \in A \,|\, am = 0 \,\forall m \in M\}$. Let N and P be submodules of M, we defined also $(N : P) = \{a \in A \,|\, aP \subseteq N\}$. Prove the following:

(i) $\mathrm{Ann}(M + N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$

(ii) $(N : P) = \mathrm{Ann}((N + P)/N)$

Exercise 3. Let A be a ring and M be a finitely generated A-module. Let $\varphi : M \twoheadrightarrow M$ be a surjective A-module homomorphism.

(i) Show that M is an $A[X]$-module via $f(X) \cdot m = f(\varphi)(m)$ for $f \in A[X]$ and $m \in M$.

(ii) Consider the ideal $I = (X) \subset A[X]$, and prove that $M = IM$

(iii) Apply the generalized Cayley-Hamilton Theorem ([AM69, Proposition 2.4]) to this situation, considering the map $\mathrm{id}_M : M \to M$ and deduce from this that φ is injective.

Exercise 4. Let A be a ring and M an A-module.

(i) Let $x_1, \dots, x_n \in M$ be elements in M. Show that the following conditions are equivalent:

  (a) For any element $m \in M$ there exist unique elements $a_i \in A$, $i = 1, \dots, n$, such that $m = \sum_{i=1}^n a_i x_i$.

  (b) $x_1, \dots, x_n$ generate M and $\sum_{i=0}^n a_i x_i = 0 \Rightarrow a_i = 0$ for all $i = 1, \dots, n$.

  (c) The A-module homomorphism defined by

$$A^n \to M, \quad (a_1, \dots, a_n) \mapsto \sum_i a_i x_i$$

  is an isomorphism.

If the equivalent conditions above are satisfied we say that the elements $x_1, \dots, x_n$ form a *free basis of length n of M*.

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on November 6th.

(ii) Assume that $M \cong A^n$ (isomorphism of A-modules). Show that if the elements $x_1, \dots, x_n$ generate M as an A-module, then they are also a free basis of M. (*Hint:* Use Exercise 3.)

(iii) Show that A^n and A^m are isomorphic as A-modules if and only if $n = m$.

(iv) Conclude from (iii) that if M is a finitely generated free A-module then any free basis of M has the same length.

By the above we can define the *rank* of a finitely generated free A-module M to be the length of a free basis, i.e. if $M \cong A^n$, then $\text{rank}(M) = n$.

Exercise 5. (i) Give an example of a ring A, an A-module M and a submodule $M' \subset M$, such that M is free of rank 1 and M' cannot be generated by less than two elements.

(ii) Give an example of a ring A, a finitely generated A-module M and a submodule $M' \subset M$ which is not finitely generated.

References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., 1969.