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# Number Theory I

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## Exercise sheet 14[1]

**Exercise 1.** Let $A$ be a ring and $M$ be an $A$-module. Prove the following:

(i) If $M$ is simple (the only sub-$A$-modules of $M$ are $M$ and 0), then any non zero element $m \in M$ generates $M$.

(ii) $M$ is simple if and only if $M \cong A/\underline{M}$ for some maximal ideal $\underline{M}$ of $A$ and if so $\underline{M} = \mathrm{Ann}(M)$.

**Exercise 2.** Let $A$ be a Noetherian ring, and $M$ a finitely generated $A$-module. Prove the following four conditions are equivalent:

(i) $M$ has finite length

(ii) $M$ is annihilated by some product of maximal ideals

(iii) every prime ideal containing $\mathrm{Ann}(M)$ is maximal

(iv) $A/\mathrm{Ann}(M)$ is Artinian

**Exercise 3.** Recall that if $A = \oplus_{i=0}^{\infty} A_i$ is a Noetherian graded ring with $A_0$ Artinian, $M = \oplus_{i=0}^{\infty} M_i$ is a finitely generated graded $A$-module, and if $\lambda$ denotes the length function defined on the set of isomorphism classes of finitely generated $A_0$-modules, then the Poincaré series of $M$ is defined to be

$$P(M, t) := \sum_{i=0}^{\infty} \lambda(M_i) t^i \in \mathbb{Z}[[t]].$$

We have proved that there exists a polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(n) = \lambda(M_n)$ for $n \in \mathbb{N}^+$ sufficiently large, and this polynomial is called the Hilbert function (or Hilbert polynomial) of $M$. Recall also that if $A$ is a Noetherian local ring with maximal ideal $\underline{M}$, and $\underline{Q}$ is $\underline{M}$-primary ideal, then there exists a polynomial $\chi^A_{\underline{Q}}(X) \in \mathbb{Q}[X]$ such that the length of $A/\underline{Q}^n$ (as an $A$-module) is equal to $\chi^A_{\underline{Q}}(n)$ for $n$ sufficiently large, and $\chi^A_{\underline{Q}}(X)$ is called the characteristic polynomial (the existence of this polynomial is proven in [AM69, Proposition 11.4]).

Now let $A = A_0[X_1, \cdots, X_s]$ be a polynomial ring in $s$ variables over an Artin ring $A_0$. This is a graded ring, and can be regarded as a graded module over itself.

[1]If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on February 5th.

(i) Calculate the Poincaré series $P(A, t)$.
(ii) Write out the Hilbert polynomial of $A$.
(iii) Let $\underline{M} \subseteq A$ be a maximal ideal containing $\underline{Q} = (X_1, \cdots, X_s)$. Calculate the characteristic polynomial $\chi_{\underline{Q}}^{A_{\underline{M}}}(X)$.

**Exercise 4.** Let $k$ be a field, and let $k[X, Y]$ be the polynomial ring in two variables equipped with the usual grading such that $\deg(X) = \deg(Y) = 1$. Consider the ideals $I = (X, Y^2)$ and $J = (X^2, Y^2)$. As a graded modules $I = \bigoplus_{n\in\mathbb{N}} I_n$ and $J = \bigoplus_{n\in\mathbb{N}} J_n$, where $I_n$ is the set of homogeneous polynomials of degree $n$ in $I$ and $J_n$ is the set of homogeneous polynomials of degree $n$ in $J$. Prove that $I$ and $J$ have different Poincaré series, but the same Hilbert polynomial.

## References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, ix+128.