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Number Theory I

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Exercise sheet 13[1]

Exercise 1. Let K be a field. A *discrete valuation* on K is a surjective map $v : K^* \to \mathbb{Z}$, where $K^* = K \setminus \{0\}$, such that

(i) $v(xy) = v(x) + v(y)$, i.e. v is a homomorphism;
(ii) $v(x + y) \geq \min(v(x), v(y))$.

An integral domain A is a *discrete valuation ring* if there is a discrete valutation on the fraction field K of A and A is the set given by the elements of $a \in K^*$ such that $v(a) \geq 0$ and $0 \in K$. Show that all ideals of A are of the form $\mathfrak{m}_k := \{y \in A \mid v(y) \geq k\}$, where $k \in \mathbb{N}$, then conclude that A is a Noetherian local ring.

Exercise 2. Let A be a Noetherian local domain with exactly two prime ideals. Let $\mathfrak{m}$ be its maximal ideal. Show that any non-trivial ideal $\mathfrak{a} \subseteq A$ is a primary ideal and it contains $\mathfrak{m}^k$ for some $k \in \mathbb{N}^+$.

Exercise 3. Let A be a Noetherian local domain with exactly two prime ideals. Let $\mathfrak{m}$ be its maximal ideal. Show that the following statements are equivalent.

(i) A is a discrete valuation ring;
(ii) A is integrally closed;
(iii) $\mathfrak{m}$ is a principal ideal;
(iv) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$;
(v) every non-zero ideal of A is a power of $\mathfrak{m}$;
(vi) there exists $x \in A$ such that every ideal of A is of the form (x^k) with $k \geq 0$.

(Hint: Do the circle i)⇒ ii)⇒ iii)⇒ iv)⇒ v)⇒ vi)⇒ i)⇒. In ii)⇒iii) you can take a nonzero element $a \in \mathfrak{m}$ and choose an integer $n \in \mathbb{N}^+$ such that $\mathfrak{m}^n \subseteq (a)$ and $\mathfrak{m}^{n-1} \not\subseteq (a)$. Take $b \in \mathfrak{m}^{n-1}$ with $b \notin (a)$, then use AM Proposition 2.4 to show that $\mathfrak{m} = (a/b)$.)

Exercise 4. Show that A is a Noetherian valuation ring if and only if A is a discrete valuation ring.

Exercise 5. Let A be a local integral domain with maximal ideal $\mathfrak{m} \neq 0$. Suppose that $\mathfrak{m}$ is principal and $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Show that A is a discrete valuation ring.

[1] If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on January 29th.