January 14, 2016

Number Theory I

Prof. H. Esnault, Dr. V. Di Proietto

Exercise sheet 12[1]

Exercise 1. Let A be a ring. Let $P_0, \dots, P_n$ be prime ideals , we say that $P_0 \subset \cdots \subset P_n$ is a chain if the inclusions are strict; the length of the chain is n. We define the dimension of A and we write $\dim(A)$ to be the supremum of the lengths of all chains of prime ideals in A. Let $A \subseteq B$ be an integral extension, prove using the "Going up" theorem that $\dim(A) = \dim(B)$.

Exercise 2. Let $A \subseteq B$, and B integral over A.

(i) If $x \in A$ is a unit in B, then it is a unit in A.
(ii) Let J_A be the Jacobson radical of A, and let J_B the Jacobson radical of B; prove that $J_A = J_B \cap A$.

Exercise 3. Let A be an integral domain, K its field of fractions. Show that the following conditions are equivalent

(i) A is a valuation ring of K
(ii) If I and J are two ideals of A, then either $I \subseteq J$ or $J \subseteq I$.

Deduce that if A is a valuation ring and P is a prime ideal of A, then A_P and A/P are valuation ring of their field of fractions.

Exercise 4. Let Γ be a totally ordered abelian group (written additively), this means an abelian group Γ with identity element given by e, equipped with a relation such that $g \leq h$ or $h \leq g$ for every $g, h \in \Gamma$ and such that $\leq$ is antisymmetric (if $g \leq h$ and $h \leq g$ then $g = h$, for $g, h \in \Gamma$) and transitive (if $g \leq h$ and $h \leq t$, then $g \leq t$ for every $g, h, t \in \Gamma$) and such that if $g, h, t \in \Gamma$, $g \leq h$ implies $g + t \leq h + t$.

Let K be a field. A valuation with values in Γ is a map $v : K^\times \to \Gamma$ such that

(i) $v(xy) = v(x) + v(y)$
(ii) $v(x + y) \geq \min(v(x), v(y))$

for all $x, y \in K^\times$. Let $\overline{\Gamma}$ be $\Gamma \cup \{\infty\}$ where ∞ is a symbol such that $\gamma \leq \infty$ for every $\gamma \in \Gamma$. We extend v to $\overline{v} : K \to \overline{\Gamma}$ setting $\overline{v}(x) = v(x)$ for every $x \in K^\times$ and $\overline{v}(0) = \infty$. Show that the set of elements $x \in K$ such that $v(x) \geq e$ is a valuation ring of K.

[1]If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on January 22th.