December 17th, 2015

Number Theory I

Prof. H. Esnault, Dr. V. Di Proietto

Exercise sheet 10[1]

Exercise 1. Let $f : A \to B$ be a map of rings. The map f is called faithfully flat if it is flat and if for any A-module M, $M \otimes_A B = 0$ implies that $M = 0$. Now suppose that f is faithfully flat. Show that if B is an Artinian ring then A is also Artinian.

Exercise 2. Let A be a commutative ring with exactly one prime ideal $\mathfrak{m}$. Show that A is Artinian if and only if $\mathfrak{m}$ is finitely generated. Give an example for a commutative ring with exactly one prime ideal which is not Artinian.

Exercise 3. Let k be a field, A a finitely generated k-algebra. Prove that the following two conditions are equivalent:

(i) A is Artinian;

(ii) A is a finitely k-algebra, i.e. A is finite dimensional as a k-vector space.

If A is not a finitely generated k-algebra, is this still true? If not, what breaks down? Argue with examples.

Exercise 4. Show that if A is a ring in which all the prime ideals are finitely generated, then A is Noetherian. (Hint: You want to show that all the ideals are finitely generated. So take the collection of ideals which are not finitely generated. Pick a maximal element I using Zorn's lemma. Then show that I has to be a prime ideal.)

[1] If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on January 8th.