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# Number Theory I

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## Exercise sheet 1[1]

**Exercise 1.** Let $A$ be a ring.

(i) Assume $a \in A$ is a nilpotent element, *i.e.* there exists $n \in \mathbb{N}$ such that $a^n = 0$. Show that $1+a$ is a unit of $A$. (*Hint:* Think about the geometric series trick!)

(ii) Deduce from (i), that if $u \in A^\times$ is a unit and $a \in A$ is nilpotent, then $u + a \in A^\times$ is a unit.

**Exercise 2.** Let $A$ be a ring and $A[X]$ be the set whose objects are $\sum_{i=0}^{\infty} a_i X^i$ where $a_i \in A$, and $a_i = 0$ for all but a finite numbers of values of $i$. Then $A[X]$ is a ring with the usual sum and multiplication between polynomials.

Let $f = a_0 + a_1 X + \ldots + a_n X^n \in A[X]$ be a polynomial with $a_i \in A$, $i = 0, \ldots, n$. Prove:

(i) $f \in (A[X])^\times \Longleftrightarrow a_0 \in A^\times$ and $a_1, \ldots, a_n$ are nilpotent.
(*Hint:* For $\Rightarrow$: If $b_0 + \ldots + b_m X^m$ is an inverse of $f$ show by induction on $r$ that $a_n^{r+1} b_{m-r} = 0$, for $r \geq 0$. Deduce that $a_n$ is nilpotent and use Ex. 1, (ii).)

(ii) $f$ is nilpotent $\Longleftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

(iii) $f$ is a zero-divisor $\Longleftrightarrow \exists\, a \in A \setminus \{0\}$ such that $af = 0$.
(*Hint:* For $\Rightarrow$: Choose $g = b_0 + \ldots + b_m X^m \neq 0$ of least degree $m$ such that $fg = 0$. Then $a_n b_m = 0$. Deduce $a_n g = 0$ and by induction $a_{n-r} g = 0$, all $0 \leq r \leq n$. Deduce the statement.)

Describe the units, nilpotent elements and zero-divisors of $A[X]$ in the case where $A$ is an integral domain.

**Exercise 3.** Let $A$ be a ring in which every element satisfies $a^n = a$ for some natural number $n > 1$ (depending on $a$). Show that every prime ideal in $A$ is maximal.

**Exercise 4.** Let $A$ be a local ring, *i.e.* a ring with exactly one maximal ideal, then if there exists $e \in A$ such that $e^2 = e$, then $e = 0$ or $e = 1$ (an element $e$ such that $e^2 = e$ is called idempotent).

**Exercise 5.** Which of the following ideals are prime, which are maximal?

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on October 23th.

(i) $(0)$, $(5)$, $(7)$, $(7365)$ in $\mathbb{Z}$
(ii) $(0)$, $(5)$, $(7)$, $(7365)$ in $\mathbb{Q}$
(iii) $(X-\lambda)$, $(X^3)$, $(\lambda)$, $(X^2-1)$ in $K[X]$, where $K$ is a field and $\lambda \in K \setminus \{0\}$
(iv) $(13)$, $(2, X^2+X+1)$, $(2, X^3+X^2+X+1)$, $(X^2+1)$ in $\mathbb{Z}[X]$
(v) $(\pi X)$, $(XY)$, $(X^2-Y^2)$ in $\mathbb{R}[X,Y]$