November 24th, 2015

Remark on Lecture of November 24th

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Theorem 1. *Let A be a ring and let I be an ideal of A. If $\sqrt{I} = M$, where M is a maximal ideal of A, then I is an M-primary ideal*

Proof. To prove that I is a primary ideal we are going to prove that in A/I every zero divisor is nilpotent. By [AM69, Proposition 1.14] we have that

$$\sqrt{I} = \bigcap_{P \text{ prime ideal} \,|\, I \subset P} P$$

Hence $M = \sqrt{I}$ is an intersection of prime ideals. The image of M in A/I, $\bar{M}$, is a maximal ideal, because the one-to-one correspondence between the ideals of A which contain I and the ideals of A/I preserves maximal and prime ideals. Hence $\bar{M} = \bigcap_{\bar{P} \text{ prime ideal of } A/I} \bar{P}$ is a maximal ideal of A/I which is an intersection of primes ideals. By [AM69, Proposition 1.11 (ii)] $\bar{M} = \bar{P}_1$ for some $\bar{P}_1$ prime ideal of A/I. We claim that in A/I there is only one prime ideal, namely $\bar{P}_1$. Indeed, suppose that there exists $\bar{P}_2$ a prime ideal of A/I, then $\bar{P}_1 = \bar{M} = \bigcap_{\bar{P} \text{ prime ideal of } A/I} \bar{P} \subset \bar{P}_2$, hence since $\bar{M}$ is maximal, then $\bar{P}_1 = \bar{P}_2$. Hence A/I is a local ring.

Let $x \in A/I$, then $x \in \bar{M}$ or $x \in (A/I)^{\times}$. Therefore if x is a zero divisor so that in particular $x \notin (A/I)^{\times}$, then $x \in \bar{M}$. But $\bar{M} = \bigcap_{\bar{P} \text{ prime ideal of } A/I} \bar{P}$ is the nilradical of A/I, *i.e* it is nilpotent as we wanted to prove. □

References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., 1969.