November 10th, 2013

Algebra I

Prof. H. Esnault

Exercise Sheet 9[1]

Exercise 9.1. Let X be a compact Hausdorff space, $\mathcal{C}(X)$ the ring of real-valued continuous functions on X. Is the zero ideal decomposable in this ring?

(*Hint:* Show that if $\mathfrak{p} \subset \mathcal{C}(X)$ is a prime ideal, then the locus $V(\mathfrak{p}) = \{x \in X \mid f(x) = 0 \quad \forall f \in \mathfrak{p}\}$ consists only of a single point of X.)

Exercise 9.2. Let A be a ring, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ that satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$. Show:

(a) $x \in A$ is a zero divisor if and only if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$. (*Hint:* The radical of $(0 : a)$ is the intersection of the minimal prime ideals containing $(0 : a)$.)

(b) If the zero ideal has a primary decomposition, then $D(A)$ is the set of associated prime ideals of 0.

(c) $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = (0)$, where $S_{\mathfrak{p}}(0)$ denotes the kernel of the homomorphism $A \to A_{\mathfrak{p}}$.

Exercise 9.3. Let $A \subset B$ be an integral ring extension. Show:

(a) If $x \in A$ is a unit in B, then it is a unit in A.

(b) Let $f : A \to F$ be a homomorphism of A into an algebraically closed field F. Then f can be extended to a homomorphism of B into F.

Exercise 9.4. Let A be an integral domain. Show the following theorem:

If A is integrally closed, then the polynomial ring $A[X]$ is integrally closed.

You can proceed as follows: Let A be a subring of an integral domain B, and let C be the integral closure of A in B.

(a) Let f, g be monic polynomials in $B[X]$ such that $fg \in C[X]$. Then f, g are in $C[X]$.
(*Hint:* Take a field containing B in which the polynomials f, g split into linear factors. Each root of f and each root of g is a root of fg, hence is integral over C. This implies that the coefficients of f and g are integral over C.)

[1] If you want your solutions of this exercise sheet to be corrected, please hand them in just before the lecture on November 17th. Questions or comments to `henrik.russell@math.fu-berlin.de` or come to office A3, 112.

(b) Prove that $C[X]$ is the integral closure of $A[X]$ in $B[X]$.
(*Hint:* If $f \in B[X]$ is integral over $A[X]$, then there is an equation $f^m + g_{m-1}f^{m-1} + \ldots + g_0 = 0$ with $g_i \in A[X]$. Let r be an integer larger than m and the degrees of $g_0, \ldots, g_{m-1}$, and let $f_1 = f - X^r$, so that $(f_1 + X^r)^m + g_{m-1}(f_1 + X^r)^{m-1} + \ldots + g_0 = 0$. This can be written in the form $f_1^m + h_{m-1}f_1^{m-1} + \ldots + h_0 = 0$ for certain $h_i \in A[X]$. Now apply step (a) to the polynomials f_1 and $f_1^{m-1} + h_{m-1}f_1^{m-2} + \ldots + h_1$.)

(c) Now let $B = K$ be the fraction field of A and assume that A is integrally closed. Prove the theorem.
(*Hint:* Use transitivity. Why is $K[X]$ integrally closed? Use step (b).)