December 3rd, 2013

# Algebra I

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## Exercise sheet 8[1]

**Exercise 1.** Let $k[x, y, z]$ be the polynomial ring in three variables over a field $k$. Let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (y, z)$, $\mathfrak{m} = (x, y, z)$, $\mathfrak{a} := \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$. What are the associated primes of $\mathfrak{a}$? Which of them are minimal and which of them are embedded?

**Exercise 2.** Let $A$ be a commutative ring, $A[x]$ be the ring of polynomials in one indeterminate over $A$. For each ideal $\mathfrak{a}$ of $A$, let $\mathfrak{a}[x]$ be the set of all polynomials in A[x] with coefficients in $\mathfrak{a}$.

(a) $\mathfrak{a}[x]$ is the ideal of $A[x]$ generated by $\mathfrak{a}$.
(b) If $\mathfrak{p}$ is a prime ideal in $A$, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.
(c) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in $A$, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$.
(d) If $\mathfrak{a} = \cap_{i=1}^{n}\mathfrak{q}_i$ is a minimal primary decomposition in $A$, then $\mathfrak{a}[x] = \cap_{i=1}^{n}\mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.
(e) If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

**Exercise 3.** Let $A$ be a commutative ring, $S$ a multiplicatively closed subset of $A$. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the inverse image of $S^{-1}\mathfrak{a}$ under the localization map $A \to S^{-1}A$. Prove that

(a) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$.
(b) $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$, where $r(\mathfrak{a})$ means the radical ideal of $\mathfrak{a}$.
(c) $S(\mathfrak{a}) = A \Leftrightarrow \mathfrak{a}$ meets $S$.
(d) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$, where $S_1, S_2$ are two multiplicatively closed subsets of $A$.
(e) If $\mathfrak{a}$ has a primary decomposition, then the set of ideals $S(\mathfrak{a})$, when $S$ runs through all multiplicatively closed subsets of $A$, is a finite set.

[1]If you want your solutions to be corrected, please hand them in just before the lecture on Dec.10th. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

**Exercise 4.** Let $A$ be a commutative ring, $\mathfrak{p}$ be a prime ideal. In the notation of the last exercise (Ex 8.3), we denote $\mathfrak{p}^{(n)} := S_{\mathfrak{p}}(\mathfrak{p}^n)$ where $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$. Prove that

(a) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;
(b) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary component;
(c) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is a $\mathfrak{p}$-primary component;
(d) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \Leftrightarrow \mathfrak{p}^n$ is $\mathfrak{p}$-primary.