November 18th, 2013

Algebra I

Prof. H. Esnault

Exercise Sheet 6[1]

Exercise 6.1. Let A be a ring.

(a) Show (or maybe recall) that the following are equivalent:
  (i) $A \cong A_1 \times A_2$, where A_1 and A_2 are non-zero rings.
  (ii) A contains an idempotent $e \neq 0, 1$.

(b) If $A \cong A_1 \times A_2$ as above, realize the factor A_1 as a localization $S_f^{-1}A$ for some $f \in A$, where $S_f = \{f^n \mid n \in \mathbb{N}\}$.

(c) Show that the prime ideals of the product ring $A \cong A_1 \times A_2$ are of the form $\mathfrak{p} \times A_2$ or $A_1 \times \mathfrak{q}$, for some $\mathfrak{p} \in \operatorname{Spec} A_1$ or $\mathfrak{q} \in \operatorname{Spec} A_2$, respectively.

Exercise 6.2. Let A be a ring.

(a) Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$.

(b) If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain? Give a proof or a counter-example.
(*Hint:* Take Exc. 6.1 into account.)

Exercise 6.3. Let A be a ring $\neq 0$.

(a) Let Σ be the set of all multiplicatively closed subsets S of A such that $0 \neq S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A \setminus S$ is a minimal prime ideal of A.

(b) Let S_0 be the set of all non-zero-divisors in A. Show:
  (i) S_0 is a saturated multiplicatively closed subset of A (this notion was introduced in Exc. 5.4). Hence the set D of zero-divisors in A is a union of prime ideals (see Exc. 5.4 (iii)). Every minimal prime ideal of A is contained in D. (*Hint:* Use (a).)

  The ring $S_0^{-1}A$ is called the *total ring of fractions* of A. Show:
  (ii) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.
  (iii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

[1]If you want your solutions of this exercise sheet to be corrected, please hand them in just before the lecture on November 26th. Questions or comments to henrik.russell@math.fu-berlin.de or come to office A3, 112.

(iv) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e. $A \to S_0^{-1}A$ is bijective).

Exercise 6.4. Let A be a ring and M an A-module. Suppose that $f_1, \dots, f_r \in A$ generate the unit ideal: $(f_1, \dots, f_r) = (1)$. Show:

(a) $(f_1^{n_1}, \dots, f_r^{n_r}) = (1)$ for all $n_1, \dots, n_r \in \mathbb{N}$.

(b) For $m \in M$, if $M \to S_{f_i}^{-1}M$ maps $m \mapsto 0$ for all $i = 1, \dots, r$, then $m = 0$.

(c) If $m_i \in S_{f_i}^{-1}M$ $(i = 1, \dots, r)$ are elements such that m_i and m_j are mapped to the same element in $S_{f_i f_j}^{-1}M$ for each pair (i, j), then there is a unique element $m \in M$ that is mapped to $m_i \in S_{f_i}^{-1}M$ for each i.