November 4th, 2013

Algebra I

Prof. H. Esnault

Exercise sheet 4[1]

Exercise 1. Prove the following statements:

(i) For any $n \in \mathbb{Z}$, there is an isomorphism of $\mathbb{Z}$-modules

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$$

(ii) If $n, m \in \mathbb{Z}$ are coprime, i.e. if they do not have a common prime divisor, then $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = 0$.

(iii) If A is a commutative ring, M an A-module, and $I \subset A$ an ideal, then prove that there is an isomorphism of A-modules

$$(A/I) \otimes_A M \cong M/IM.$$

Exercise 2. Let A be a commutative ring.

(i) If A is a commutative ring and M, N two finitely generated A-modules, show that $M \otimes_A N$ is finitely generated.

(ii) Recall the definition of the *rank* of a finitely generated free A-module from the previous exercise sheet.

If M, N are free A-modules of finite rank, show that $M \otimes_A N$ is a free A-module of finite rank. Compute $\text{rank}(M \otimes_A N)$ in terms of $\text{rank}(M)$ and $\text{rank}(N)$.

(iii) Let A be a commutative local ring, and M, N finitely generated A-modules. If $M \otimes_A N = 0$, then either $M = 0$ or $N = 0$. *(Hint: First prove this if A is a field. If B is an A-algebra, prove that $(M \otimes_A B) \otimes_B (N \otimes_A B) = (M \otimes_A N) \otimes_A B$. Then use Nakayama's Lemma.)*

Exercise 3. Let A be a commutative ring and P an A-module. Show that the following properties are equivalent:

(i) For any short exact sequence

$$0 \to L \to M \to N \to 0$$

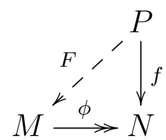
the associated sequence

$$0 \to \text{Hom}_A(P, L) \to \text{Hom}_A(P, M) \to \text{Hom}_A(P, N) \to 0$$

is also exact.

[1] If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on November 12. Questions or comments to kindler@math.fu-berlin.de or come to A3, Room 112A.

(ii) For any surjective morphism $\phi : M \to N$ of A-modules, and any A-linear morphism $f : P \to N$, there exists a morphism $F : P \to N$ lifting f, i.e. such that the diagram



commutes.

(iii) Every short exact sequence of A-modules

$$0 \to L \to M \xrightarrow{\tau} P \to 0$$

splits, i.e. there exists an A-module homomorphism $\sigma : P \to M$, such that $\tau \circ \sigma = \mathrm{id}_P$.

(iv) P is a direct summand of a free A-module.

If P satisfies the above equivalent properties, then P is called *projective.*

Exercise 4. Use Nakayama's Lemma to show that if A is a local ring, i.e. if A has precisely one maximal ideal, and if P is a finitely generated projective A-module, then P is free of finite rank.