October 29th, 2013

Algebra I

Prof. H. Esnault

Exercise sheet 3[1]

Exercise 1. Let A be a ring and M a finitely generated A-module. Show that any surjective A-module homomorphism $\varphi : M \twoheadrightarrow M$ is automatically an isomorphism. (*Hint:* View M as an $A[X]$-module via $f(X).m = f(\varphi)(m)$ for $f \in A[X]$ and $m \in M$. Consider the ideal $I = (X) \subset A[X]$. Now apply the Theorem of Cayley-Hamilton (Thm 2 in §8) to the situation $\mathrm{id}_M : M \to M = I.M$ and conclude.)

Exercise 2. Let A be a ring and M an A-module.

(i) Let $x_1, \dots, x_n \in M$ be elements in M. Show that the following conditions are equivalent:
  (a) For any element $m \in M$ there exist unique elements $a_i \in A$, $i = 1, \dots, n$, such that $m = \sum_{i=1}^n a_i x_i$.
  (b) $x_1, \dots, x_n$ generate M and $\sum_{i=0}^n a_i x_i = 0 \Rightarrow a_i = 0$ for all $i = 1, \dots, n$.
  (c) The A-module homomorphism defined by

$$A^n \to M, \quad (a_1, \dots, a_n) \mapsto \sum_i a_i x_i$$

  is an isomorphism.

If the equivalent conditions above are satisfied we say that the elements $x_1, \dots, x_n$ form a *free basis of length n of M*.

(ii) Assume that $M \cong A^n$ (isomorphism of A-modules). Show that if the elements $x_1, \dots, x_n$ generate M as an A-module, then they are also a free basis of M. (*Hint:* Use Exercise 1.)
(iii) Show that A^n and A^m are isomorphic as A-modules if and only if $n = m$.
(iv) Conclude from (iii) that if M is a finitely generated free A-module then any free basis of M has the same length.

By the above we can define the *rank* of a finitely generated free A-module M to be the length of a free basis, i.e. if $M \cong A^n$, then $\mathrm{rank}(M) = n$.

[1] If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on October 29. Questions or comments to kay.ruelling@fu-berlin.de or come to A3, Room 108.

Exercise 3. Give an example of a ring A, an A-module M and a submodule $M' \subset M$, such that M is free of rank 1 and M' cannot be generated by less than two elements.

Exercise 4. Let $A = A_1 \times \ldots \times A_n$ be the product of n rings.

(i) Show that any A module M is isomorphic as an A-module to a direct sum $\oplus_{i=1}^n M_i$ of A_i-modules M_i, which are viewed as A-modules via the projection to the i-th coordinate $A \to A_i$. (*Hint:* Set $M_i := e_i M$ where $e_i = (0, \ldots, 1, 0 \ldots, 0) \in A$, with the 1 in the i-th place.)

(ii) In the above situation, show that M is a free A-module of rank n if and only if all the M_i are free A_i-modules of the same rank n.

Exercise 5. Let A be a ring, M an A-module and $I \subset A$ a *nilpotent ideal*, i.e. an ideal such that there exists a natural number n with $I^n = 0$. Show that $I.M = M$ implies $M = 0$. (Notice that we do not assume M to be finitely generated.)