October 22th, 2013

Algebra I

Prof. H. Esnault

Exercise sheet 2[1]

2.1. Recall (from Algebra and Number theory) that a ring A is called a *unique factorization domain (UFD)* if every element $a \in A$ which is neither zero nor a unit can be written as a product of prime elements $a = p_1 \cdots p_n$. (An element $p \in A$ is *prime* iff the ideal $(p) \subset A$ is a prime ideal.)

Exercise 1. The following is a list of elementary properties of UFD's, which you probably already know. You can prove them again if you want (but you can also skip them). Then you can use them in the other exercises.

(i) Let A be an integral domain and assume we can write $a \in A$ as a product of prime elements $a = p_1 \cdots p_n$. Then this presentation is unique up to permutation and multiplication with units, i.e. if $a = p'_1 \cdots p'_n$ with prime elements $p'_i \in A$, then there exists a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and units $u_i \in A^\times$ such that $p'_i = u_i p_{\sigma(i)}$.

Recall that two prime elements $p, p' \in A$ are *equivalent* if there exists a unit $u \in A^\times$ with $p' = up$. The above result in particular implies, that if A is a UFD, I is the set of equivalence classes of prime elements in A and for each $i \in I$ we pick a representative $p_i \in i$, then any element $a \in A \setminus \{0\}$ can be uniquely written in the form

$$a = u \prod_{i \in I} p_i^{n_i},$$

with $u \in A^\times$ and $n_i \in \mathbb{N}$, $i \in I$, all but finitely many $n_i = 0$.

(ii) Let A be an integral domain. Recall that an element $a \in A \setminus \{0\}$ is *irreducible* if it is not a unit and if we can write $a = bc$, then either $b \in A^\times$ or $c \in A^\times$.

Show that a prime element in A is always irreducible. Show that if A is a UFD, then an irreducible element is also prime.

(iii) Show that a PID is UFD. (*Hint:* First show that in a PID any ascending chain of ideals $I_1 \subset I_2 \subset \ldots$ becomes stationary,

[1]If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on October 29. Questions or comments to kay.ruelling@fu-berlin.de or come to A3, Room 108.

i.e. we have $I_n = I_{n+1}$ for all n large enough. Then show that in a PID any non-zero element a which is not a unit can be written as $a = pa_1$ with p a prime. Continue with a_1 and so on and deduce the statement.)

(iv) (Gauss Lemma) Let A be a UFD. We say a polynomial $f = \sum_{i=0}^n a_i X^i \in A[X]$ is *primitive* if the coefficients $a_1, \dots, a_n$ are not divisible by a common prime element.

Show that if $f, g \in A[X]$ are primitive then so is fg.

Exercise 2. Let A be an integral domain.

(i) We say $(a, b), (a', b') \in A \times A \setminus \{0\}$ are equivalent if $ab' = a'b$. Denote by K the set of equivalence classes of the pairs (a, b). We write $\frac{a}{b} \in K$ for the equivalence class of (a, b). Show that the operations

$$\frac{a}{b} + \frac{a'}{b'} := \frac{ab' + ba'}{bb'}, \quad \frac{a}{b} \cdot \frac{a'}{b'} := \frac{aa'}{bb'}, \quad -\frac{a}{b} := \frac{-a}{b}$$

are well-defined and give K the structure of a field with neutral elements $0_K = \frac{0}{1}$ and $1_K = \frac{1}{1}$. Further show that $A \to K$, $a \mapsto \frac{a}{1}$ is an injective ring homomorphism. (In the following we will view $A \subset K$ as a subring and identify the elements a and $\frac{a}{1}$.)

(ii) Show that any $f \in A[X] \setminus \{0\}$ can be written as $f = af_0$ where $f_0 \in A[X]$ is primitive (in the sense of Ex.1, (v)) and $a \in A \setminus \{0\}$ is not a unit.

(iii) Show that if $p \in A$ is a prime element in A then it is also a prime element in $A[X]$.

(iv) Assume that A is a UFD. Show that if $f \in A[X]$ is primitive and its image in $K[X]$ is prime, then $f \in A[X]$ is also prime. (*Hint:* Use (v) of Ex. 1 and (ii) above.)

(v) Deduce from (ii), (iii) and (iv) above that if A is a UFD then so is $A[X]$. (*Hint:* Notice that we know that $K[X]$ is a PID and hence by Ex. 1, (iv) also a UFD.)

Remark 1. Ex. 1, (iv) and Ex. 2(v) together imply that $\mathbb{Z}[X_1, \dots, X_n]$ and $K[X_1, \dots, X_n]$ (K a field) are UFD's.

Exercise 3.

- Show that $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible but not prime. Hence $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. (*Hint:* To show that 2 is not prime try to factor $6 \in \mathbb{Z}[\sqrt{-5}]$ in two different ways.)
- By Exercise 4 on sheet 1 the ring $A = K[X, Y]/(Y^2 - X^3)$ is a domain (K a field). Show that $y = \bar{Y} \in A$ is irreducible but not prime. (Thus A is not a UFD.)

Exercise 4. For a ring A we denote by $\operatorname{Spec} A$ the set of prime ideals in A, it is called the *spectrum of* A. Describe the following sets

$$\operatorname{Spec}\mathbb{R}, \quad \operatorname{Spec}\mathbb{Z}, \quad \operatorname{Spec}\mathbb{C}[X], \quad \operatorname{Spec}\mathbb{R}[X], \quad \operatorname{Spec}\mathbb{Z}[X],$$
$$\operatorname{Spec}(K[X]/(X^2)), \quad \operatorname{Spec}(\mathbb{C}[X,Y]/(XY)).$$

Exercise 5. Let $f : A \to B$ be a ring homomorphism. From the lecture we know that this induces a map $f^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$, $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$.

(i) Let $A_1, \ldots, A_n$ be rings, denote by $A = A_1 \times \ldots \times A_n$ their product and by $\pi_i : A \to A_i$, $(a_1, \ldots, a_n) \mapsto a_i$, $i = 1 \ldots, n$, the projection maps.

Show that $\pi_i^{-1} : \operatorname{Spec} A_i \to \operatorname{Spec} A$ maps bijectively onto $\pi_i^{-1}(\operatorname{Spec} A_i)$ and that we have the following decomposition of $\operatorname{Spec} A$ into disjoint sets

$$\operatorname{Spec} A = \pi_1^{-1}(\operatorname{Spec} A_1) \sqcup \ldots \sqcup \pi_n^{-1}(\operatorname{Spec} A_n) \xleftarrow{\text{bij.}} \operatorname{Spec} A_1 \sqcup \ldots \sqcup \operatorname{Spec} A_n.$$

(ii) Let $\pi : A \to A_{\text{red}} = A/\text{nil(A)}$ be the canonical surjection. Show that $\pi^{-1} : \operatorname{Spec} A_{\text{red}} \to \operatorname{Spec} A$ is bijective.