FREIE UNIVERSITÄT BERLIN
SEMINAR WS 15/16

Analytic Methods in Number Theory

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INTRODUCTION

In this seminar we are going to discuss the applications of analytic methods to number theory. The first aim the seminar is to prove *Dirichlet's theorem on arithmetic progressions* which states that: given any two positive coprime integers a, m, the set of numbers $\{a + mk|k \in \mathbb{N}\}$, i.e. the set $\{a, a + m, a + 2m, a + 3m, \cdots\}$, contains infinitely many prime numbers. The proof resorts to Dirichlet series, Riemann Zeta functions, Dirichlet L-function which are very basic tools in the study of analytic number theory. Then we are going to study *Modular Forms*. Modular forms are holomorphic functions on the upper half plan satisfying certain conditions with respect to some group actions. It is another very important tool to number theory. The prerequests for this seminar are rather few. A certain familiarity with undergraduate level real and complex analysis is enough.

EINFÜHRUNG

In diesem Seminar werden wir die Anwandung der analytische Methode auf Zahlentheorie studieren. Zunächst werden wir den *Dirichletscher Primzahlsatz* beweisen, der besagt: Es sei m eine natürliche Zahl und a eine zu m teilerfremde natürliche Zahl. Dann enthält die arithmetische Folge $a, a + m, a + 2m, a + 3m, \cdots$ unendlich viele Primzahlen. Im Ablauf des Beweis lernt mann Dirichletreihe, Riemann Zeta funktion, Dirichlet L-funktion, die mann als Spache der analytischen Zahlentheorie denkt. Anschlissend werden wir uns mit Modulform beschäftigen. Eine Modulform ist eine Holomorphe Funktion auf der oberen Halbebene mit ein paar Axiome. Modulform ist auch ein sehr wichtig Tool von analytischen Zahlentheorie.

Voraussetzung zur Teilnahme sind sehr wenig. Es reicht mit grundlegende Kenntnisse über reelle und komplexe Analysis. Es kann sowohl Proseminar- als auch Seminarscheine erwerben werden.

1 Dirichlet's theorem on arithmetic progressions (14/10/2015)

Aim of the talk: Give an overview on this subject.

Details: Please include the following:

- State Dirichlet's theorem on arithmetic progressions and sketch the proof;
- State Euclid's theorem and give a proof which is independent of Dirichlet's theorem;
- The literature of Dirichlet's theorem.

2 Group characters (I) (21/10/2015)

Aim of the talk: Introduce the notion of characters for finite abelian groups.

Details: Explain the following:

- Recall the notion of groups, group homomorphisms, abelian groups, finite groups, group orders, and give examples. In particular explain why the non-zero complex numbers $\mathbb{C}^*$ form a group;
- Introduce the notion of character of a finite abelian group and give examples [Ser, Ch VI, §1, Definition 1., Example.];
- Convince us why this notion is "good" by showing that the functor "taking character" is exact ([Ser, Ch VI, §1, Proposition 1.]), self dual (([Ser, Ch VI, §1, Proposition 3.])), and preserves the group orders ([Ser, Ch VI, §1, Proposition 2.]).

3 Group characters (II) (28/10/2015)

Aim of the talk: Complete the introduction of group characters.

Details: Introduce the following:

- Group characters has orthogonal relations [Ser, Ch VI, §1, 1.2.];
- Introduce a very important example of group characters – Modular characters [Ser, Ch VI, §1, 1.3.] and give examples.
- Prove [Ser, Ch VI, §1, 1.3., Proposition 5.].

4 Dirichlet series (04/11/2015)

Aim of the talk: Introduce the notion of Dirichlet series.

DETAILS:

- First introduce Dirichlet series and give examples;
- Discuss the set of convergence of Dirichlet series following [Ser, Ch VI, §2.].

5 THE ZETA FUNCTION (11/11/2015)

Aim of the talk: STUDY THE CONVERGENCY OF A SPECIAL DIRICHLET SERIES — THE ZETA FUNCTION.

DETAILS: The main object of study is [Ser, Ch VI, §3, Proposition 10.] and its corollaries.

6 THE *L*-FUNCTIONS (18/11/2015)

Aim of the talk: STUDY THE $L-$FUNCTIONS.

DETAILS: Just follow [Ser, Ch VI, §3, 3.3., 3.4.].

7 THE DIRICHLET THEOREM (25/11/2015)

Aim of the talk: USING THE TOOLS DEVELOPED SO FAR TO PROVE THE MAIN THEOREM

DETAILS:

- Introduce the notion of density and reformulate Dirichlet theorem in a more precise way using densitiy;
- Prove the main theoerem [Ser, Ch VI, §4, Theorem 2.].
- Apply the main theorem to a case of "local global principle" follow [Ser, Ch VI, §4, 4.4.].

8 MODULAR GROUPS (02/12/2015)

Aim of the talk: STUDY THE MODULAR GROUP AND ITS FUNDAMENTAL DOMAIN.

DETAILS: Explain following things from [Ser, Chapter VII, §1].

- The notion of modular group: Definition 1.
- The notion and main properties of the fundamental domain of the modular group: Theorem 1. and its Corollary.
- The generators of the modular group: Theorem 2.

9 MODULAR FUNCTIONS (09/12/2015)

Aim of the talk: INTRODUCE THE NOTION OF MODULAR FUNCTIONS AND GIVE EXAMPLES.

DETAILS:

- Introduce the notion of weakly modular function, modular function, modular form, cusp form.
- Interpret modular functions in terms of lattice functions [Ser, Ch VII, §2, 2.2.].
- Introduce Eisenstein series as an example of modular functions.

10 THE ZEROS AND POLES OF A MODULAR FUNCTION (16/12/2015)

Aim of the talk: STUDY THE ZEROS AND POLES OF A MODULAR FUNCTION.

DETAILS:

- Introduce the notion of zeros and poles of a modular function.
- Prove the formula for zeros and poles of a modular function following [Ser, Ch VII, §3, Theorem 3.]. Please explain why the infinite sum makes sense.

11 THE SPACE OF MODULAR FORMS AND MODULAR INVARIANT (06/01/2016)

Aim of the talk: STUDY THE ALGEBRAIC STRUCTURE OF MODULAR FORMS OF A FIXED WEIGHT AND INTRODUCE THE MODULAR INVARIANT.

DETAILS:

- The algebraic structure of modular forms of weight k is analysed in [Ser, Ch VII, §3, 3.2.].
- The modular invariant is studied in [Ser, Ch VII, §3, 3.3.]. Note that the remarks at the end of the subsection are very useful.

12 SERIES EXPANSIONS (13/01/2016)

Aim of the talk: STUDY SERIES EXPANSIONS OF CERTAIN MODULAR FORMS.

DETAILS: Follow [Ser, Ch VII, §4] closely.

13 HECKE OPERATORS (I) (20/01/2016)

Aim of the talk: DEFINE HECKE OPERATOR AND STUDY ITS ACTION ON MODULAR FUNCTIONS.

DETAILS:

- Define the Hecke operator $T(n)$ as a correspondence on the set of lattices of $\mathbb{C}$, and prove its elementary properties: [Ser, Ch VII, §5, 5.1.];

- Define the action of $T(n)$ on functions of weight $2k$, and futher study the induced action of $T(n)$ on modular functions of weight $2k$: [Ser, Ch VII, §5, 5.1., 5.3.].

14 Hecke operators (II) (27/01/2016)

Aim of the talk: Introduce eigenfunctions of the Hecke operators.

Details:

- Define the eigenfunctions of $T(n)$ and show that they are uniquely determined by the eigenvalues after normalization: [Ser, Ch VII, §5, 5.4.];
- Take Eisenstein series and δ function as examples of eigenfunctions of $T(n)$: [Ser, Ch VII, §5, 5.5.];
- Give a survey on the related topics following [Ser, Ch VII, §5, 5.6.].

15 Theta functions (I) (03/02/2016)

Aim of the talk: Lay the background for the study of theta functions.

Details:

- Recall the notion of invariant measure on a finite real vector space, the notion of being a rapidly decreasing smooth function on a real vector space... in order to understand [Ser, Chapter VII, §6, 6.1.];
- Recall the notion of quadratic forms and prove [Ser, Chapter VII, §6, Proposition 16.];
- Follow [Ser, Chapter VII, §6, 6.3.-6.4.] closely.

16 Theta functions (II) (10/02/2016)

Aim of the talk: Introduce the notion of theta functions and gvie examples.

Details: Follow [Ser, Chapter VII, §6, 6.5.-6.7.] closely.

References

[Ser] J. -P. Serre, *A Course in Arithmetic*, GTM 7, Springer-Verlag, 1973.