Jan. 10, 2019

Addendum to the course on Jan. 10th, 2019

Dear students,

Let $A = \oplus_{\mathbb{N}} \mathbb{F}_p \subset B = \prod_{\mathbb{N}} \mathbb{F}_p$.

Claim 1. There is a subgroup $C \subset B$, with $A \subset C$, $C \neq B$, and B/C finite.

Proof. We think of B not only as a group, but as a ring. Then A is an ideal. By the axiom of choice, it is contained in a maximal ideal $\mathfrak{m}$ of B. Let $k = B/\mathfrak{m}$ be the residue field. It is a characteristic p field. On the other hand, B is endowed with the Frobenius homomorphims (of rings) $F : B \to B$ defined by $x = (x_1, x_2, \ldots) \mapsto F(x) = (x_1^p, x_2^p, \ldots)$. As $x^p = x$ on $\mathbb{F}_p$, F is the identity. So it restricts to the identity on $\mathfrak{m}$, and therefore its residue $\bar{F}$ on k is the identity on k. On the other hand, $\bar{F}$ is the Frobenius of k. Thus $k = \mathbb{F}_p$. Set $C = \mathfrak{m}$. This solves the problem.

$\square$

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