Discussion 1.

(1) Recall that tensor product commutes with direct sums. It follows that if $M = R^{\oplus m}$ and $N = R^{\oplus n}$ we have

$$M \otimes_R N = (R^{\oplus m}) \otimes_R (R^{\oplus n}) = ((R \otimes_R R)^{\oplus m})^{\oplus n} \cong R^{\oplus mn}.$$

In particular, for vector spaces V, W over a field k of dimensions m and n we have a (noncanonical) isomorphism $V \otimes_k W \cong k^{mn}$. The isomorphism becomes canonical if we choose bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ for the vector spaces, as $\{v_i \otimes w_j\}$ is then basis for $V \otimes_k W$.

(2) Let k be a field, let X be a set, and consider the set of morphisms of sets[1] $R = \hom(X, k)$. Notice that this has a canonical structure of a ring; $\hom(X, k) = \prod_{x \in X} k$. Now suppose that we have a collection of k-vector spaces $\{E_x\}_{x \in X}$ indexed by some set X. Then the set[2]

$$M = \{s : X \to \amalg_{x \in X} E_x : s(x) \in E_x \ \forall\ x \in X\} \cong \prod_{x \in X} E_x$$

is an R-module: Given a function $f : X \to k$ and a section $s \in M$, we obtain a new section $x \mapsto f(x)s(x)$. Similarly, we can add sections pointwise. If we have another collection $\{F_x\}_{x \in X}$ with associated R-module N, then we can consider $M \otimes_R N$. This is nothing other than the module associated to the collection $\{E_x \otimes_k F_x\}_{x \in X}$.

The point of all this is that if we think of an arbitrary ring R as the ring of some kind of functions from a set X to a field, then a module can be thought of as a collection of vector spaces indexed by X. Then tensor product happens pointwise to each vector space.

Flatness I

Recall that if M is a module and $\alpha : N \to N'$ a morphism of modules, we sometimes write $M \otimes \alpha$ instead of $\mathrm{id}_M \otimes \alpha : M \otimes N \to M \otimes N'$.

Definition 2. Let R be a ring. An R-module M is said to be *flat* if for every injective R-module homomorphisms $\alpha : N \to N'$ the homomorphism of modules $M \otimes_R \alpha : M \otimes_R N \to M \otimes_R N'$ is also injective. A module M is said to be *faithfully flat* if it is flat, and for every homomorphism $\alpha : N \to N'$, we have $M \otimes_R \alpha = 0$ implies $\alpha = 0$.

Lemma 3. *Let M be an R-module. The following are equivalent.*

(a) M is faithfully flat.
(b) A morphism $\alpha : N \to N'$ is an injection if and only if $M \otimes \alpha$ is an injection.
(c) M is flat and for any R-module N, we have $M \otimes N \cong 0$ implies $N \cong 0$.
(d) M is flat and for any R-module homomorphism $\alpha : N \to N'$, we have $M \otimes \alpha$ is an isomorphism if and only if α is an isomorphism.

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[1] If X has some extra structure, we might consider a subset of R. For example, if X is a topological space, we might consider the set of continuous functions to $\mathbb{C}$, or if X is a smooth complex manifold, we might consider the set of analytic functions to $\mathbb{C}$, or if $X \subseteq \mathbb{C}^n$, we might consider the set of those functions of the form $x \mapsto f(x)/g(x)$ for some polynomials $f, g \in \mathbb{C}[t_1, \dots, t_n]$.

[2] If X and the E_x have some extra structure, we might consider a subset of M. For example, if X is a smooth complex manifold and E_x is the tangent space $T_x X$ at x, then $\amalg E_x$ also has a structure of smooth complex manifold and we might consider only those $s : X \to TX$ which are analytic.

Proof. (a) implies (b). Consider the kernel $K \to N$. Since M is flat, $M \otimes K \to M \otimes N \to M \otimes N'$ is exact, so if $M \otimes \alpha : M \otimes N \to M \otimes N'$ is a monomorphism, then $M \otimes K \to M \otimes N$ is the zero map. But by since M is faithfully flat, this implies that $K \to N$ is the zero map, in other words, that $\alpha : N \to N'$ is injective.

(b) implies (c). Suppose that (b) is true. From the definition of flat, we see that M is flat, so it suffices to show that if $M \otimes N \cong 0$ then $N \cong 0$. Consider the canonical morphism $\alpha : N \to 0$. If $M \otimes N \cong 0$, then $M \otimes \alpha$ is a monomorphism, so since (b) holds, it follows that $\alpha : N \to 0$ is a monomorphism. In other words, $N \cong 0$.

(c) implies (d). Let $\alpha : N \to N'$ be a homomorphism such that $M \otimes \alpha$ is an isomorphism, and consider the kernel K and cokernel C of α. Since M is flat,

$$0 \to M \otimes K \to M \otimes N \overset{M \otimes \alpha}{\to} M \otimes N' \to M \otimes C \to 0 \tag{1}$$

is exact. But $M \otimes \alpha$ is an isomorphism, so $M \otimes K$ and $M \otimes C$ are zero. But by the hypothesis (c), this implies that K and C are zero. Hence, α is an isomorphism.

(d) implies (a). Suppose (d) is true, $\alpha : N \to N'$ is a morphism such that $M \otimes \alpha = 0$, and consider the kernel K and cokernel C of α. Since M is flat, (1) is exact, but if $M \otimes \alpha$ is zero, this implies that $M \otimes K \cong M \otimes N$ and $M \otimes N' \cong M \otimes C$. Then since (d) holds, we can deduce that $K \cong N$ and $N' \cong C$. In other words, $\alpha = 0$. □

Lemma 4 ([AK2017, Lemma 9.5])**.** *A direct sum $M = \oplus_{\lambda \in \Lambda} M_\lambda$ is flat if and only if every M_λ is flat. Furthermore, M is faithfully flat if every M_λ is flat, and some M_λ is faithfully flat.*

Proof. See the book. □

Proposition 5 ([AK2017, Proposition 9.6])**.** *Every nonzero free module is faithfully flat. Every projective module is flat.*

Proof. See the book. □

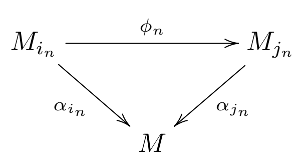
Colimits

Let R be a ring. Consider a set

$$\Phi = \{M_i, \phi_n : M_{i_n} \to M_{j_n} \mid i \in I, n \in N\} \tag{2}$$

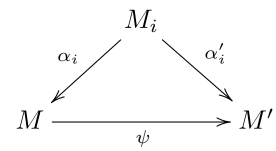
consisting of a collection of R-modules M_i indexed by a set I, and a collection of morphisms ϕ_n between them, indexed by a set N.

Definition 6. A *cone* of the system Φ is a module M equipped with a collection of morphisms $\{\alpha_i : M_i \to M\}_{i \in I}$ such that for every $n \in N$ we have $\alpha_{j_n} \circ \phi_n = \alpha_{i_n}$. That is, the triangles



are commutative for all $n \in N$. We often just write M without explicitly mentioning the α_i.

A *morphism of cones* $\{\alpha_i : M_i \to M\}_{i \in I} \to \{\alpha'_i : M_i \to M'\}_{i \in I}$ is a morphism of modules $\psi : M \to M'$ such that $\psi \circ \alpha_i = \alpha'_i$ for all $i \in I$. That is, the triangles



are commutative for all $i \in I$.

A *colimit* or *direct limit* of a system Φ, written $\varinjlim M_i$, is a universal cone. That is, it is a cone M with the property that for any other cone M', there exists a unique morphism of cones $M \to M'$.

Example 7.

(a) 0 is the colimit of the empty set.
(b) $\oplus_{i \in I} M_i$ is the colimit of the system $\{M_i \mid i \in I\}$ which has $N = \varnothing$.
(c) If $N \subseteq M$, then M/N is the colimit of the system containing the inclusion $N \hookrightarrow M$, and the zero morphism $N \stackrel{0}{\to} M$.
(d) If $\{M_i \hookrightarrow M, M_{i_n} \hookrightarrow M_{j_n} \hookrightarrow M\}$ is a collection of submodules of a module, and their inclusions in each other, then the colimit is the submodule $\langle M_i \rangle$ generated by them.

In general we can construct the colimit as follows: Consider the sum $\oplus_{i \in I} M_i$. For every $\phi_n : M_{i_n} \to M_{j_n}$, and $m \in M_{i_n}$ we have the element $m - \phi_n(m)$ in $\oplus_{i \in I} M_i$. The colimit is the quotient of the sum $\oplus_{i \in I} M_i$ by the submodule generated by elements of this form.

Definition 8. A system Φ as above is said to be *filtered* if

(1) For every M_i, M_j, there exists some M_k, and morphisms $\phi_n : M_i \to M_k$ and $\phi_m : M_j \to M_k$ in the system.
(2) For every pair of parallel morphisms $M_i \underset{\phi_m}{\overset{\phi_n}{\rightrightarrows}} M_j$ in the system, there is a morphism $\phi_\ell : M_j \to M_k$ such that $\phi_\ell \circ \phi_n = \phi_m$.

The colimit of a filtered system is called a filtered colimit.

Example 9. Let R be a domain (such as $\mathbb{Z}$) and consider the system $\{M_i, \phi_n : i \in I, n \in N\}$ such that $I = R$, $N = R \times (R \backslash \{0\})$, $M_i = R$ (for all i), and $\phi_{s,t} : R \to R : x \mapsto tx$ is multiplication by t. Then the colimit of this system is $\mathrm{Frac}(R)$. The $\alpha_s : M_s \to M$ of the sth copy of R is $R \to \mathrm{Frac}(R) : x \mapsto x/s$. This is a filtered colimit of R-modules.

When the system is filtered, there is an easier construction of the colimit. Consider the disjoint union of sets $\amalg_{i \in I} M_i$. Define an equivalence relation by $m_i \in M_i$ is equivalent to $m_j \in M_j$ if there exists a M_k, and morphisms $\phi_m : M_i \to M_k$ and $\phi_n : M_j \to M_k$ such that $\phi_m(m_i) = \phi_n(m_j)$. Then this is an equivalence relation, and the colimit of the filtered system is the quotient $\amalg_{i \in I} M_i / \sim$.

Flatness II

Proposition 10 ([AK2017, Proposition 9.9)**.** *A filtered direct limit of flat modules is flat.*

Proof. Let $\{M_i, \phi_n\}$ be a filtered system of flat modules and suppose $\beta : N \to N'$ is an injection. Then we have induced morphisms $M_i \otimes \beta : M_i \otimes N \to M_i \otimes N'$ compatible with the ϕ_n in the sense that the square

$$\begin{array}{ccc} M_{i_n} \otimes N & \xrightarrow{\phi_n \otimes N} & M_{j_n} \otimes N \\ {\scriptstyle M_{i_n}\otimes\beta}\downarrow & & \downarrow{\scriptstyle M_{j_n}\otimes\beta} \\ M_{i_n} \otimes N' & \xrightarrow[\phi_n \otimes N']{} & M_{j_n} \otimes N' \end{array}$$

commutes for every $n \in N$. As each M_i is flat, each $M_i \otimes \beta$ is injective, and it suffices to show that filtered colimits preserve injections.

Suppose that we have two systems $\{M_i, \phi_n\}_{i \in I, n \in N}$ and $\{M'_i, \phi'_n\}_{i \in I, n \in N}$ with the same indexing sets, and suppose that $\{\beta_i : M_i \to M'_i\}_{i \in I}$ is a set of injections compatible with the ϕ_n in the sense that $\beta_{j_n}\phi_n = \phi'_n\beta_{i_n}$ for all $n \in N$. Then we have an induced map of disjoint unions $\amalg M_i \to \amalg M'_i$ compatible with the equivalence relation used to define the filtered colimit, and so we get an induced map $\varinjlim M_i \to \varinjlim M'_i$. Let m be in the kernel of this map of colimits. Choose a representative $m_i \in M_i$ of $m \in \varinjlim M_i$. Since m is sent to zero, $\beta_i(m_i)$ is equivalent to zero. That is, there exists a $\phi'_n : M'_{i_n} \to M'_{j_n}$ such that $\phi'_n\beta_i(m) = 0$. But we have $\phi'_n\beta_{i_n}(m_i) = \beta_{j_n}\phi_n(m_i)$, and β_j is injective, so $\phi_n(m_i) = 0$. That is, m_i is equivalent to zero, or in other words, $m = 0$. □

Let M be an R-module, and consider the system with indexing sets $I = \{\alpha_i : R^{m_i} \to M\}$ (so the set of all maps to M from finite rank free modules), and $N = \{\phi_n : R^{m_{i_n}} \to R^{m_{j_n}} \mid \alpha_{j_n}\phi_n = \alpha_{i_n}\}$ (so the set of morphisms between the free modules compatible with the α's), and with $M_i = R^{m_i}$, and the obvious ϕ_n's.

Proposition 11 ([AK2017, Proposition 9.12])**.** *The system just described above has M as its colimit.*

Proof. See the book. □

Theorem 12 (Lazard, [AK2017, Theorem 9.13])**.** *The following are equivalent.*

- *M is flat.*
- *M is a filtered colimit of free modules of finite rank.*

Proof. See the book. □