October 17, 2017

Addenda to the course on October 17, 2017

In the sequel, all rings have a unit $1 \neq 0$.

Lemma 1. *Let R be a ring. Then an ideal $\mathfrak{p}$ is a prime ideal if and only if $S := R \setminus \mathfrak{p}$ is a multiplicative set.*

Recall

Definition 2. *An ideal $\mathfrak{p} \subset R$ is a prime ideal if and only iff*

i) $\forall x, y \in R, xy \in \mathfrak{p}, y \notin \mathfrak{p} \implies x \in \mathfrak{p}$;

ii) $\mathfrak{p} \subsetneqq R$.

A subset $S \subset R$ is a multiplicative subset iff

a) $a, b \in S \implies ab \in S$;

b) $1 \in S$.

Proof of the Lemma. i) is trivially equivalent to a). As for ii) equivalent to b): $\mathfrak{p} \subsetneqq R$ iff $\mathfrak{p} \neq \langle 1 \rangle$ iff $1 \notin \mathfrak{p}$ iff $1 \in S$.

□

In the course, I had not written b). This is now corrected.

Next here is a counter example to Proposition (2.7) in [AK, Aug. 6, 2017], in case $\varphi : R \to R'$ is not surjective. Let $R = \mathbb{Z}$, $R' = \mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, $\varphi : R \to R', x \mapsto (x, \bar{x})$ where $\bar{x} \in \mathbb{Z}/8\mathbb{Z}$ is the residue class of x. Then $I' = \langle (0, \bar{4}) \rangle \subset R'$ is not prime as $(0, \bar{2})(0, \bar{2}) = (0, \bar{4}) \in I'$ while $(0, \bar{2}) \notin I'$, yet $\varphi^{-1}(I') = \{x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x, \bar{x}) = (0, \bar{4}\bar{y})\} = \langle 0 \rangle$. As $\mathbb{Z}$ is a domain, $\langle 0 \rangle \subset \mathbb{Z}$ is a prime ideal.