June 9th, 2016

Number Theory II

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Exercise sheet 8[1]

Exercise 1. Let $\alpha = \sqrt[3]{2} \in \mathbb{R}$ and consider $K = \mathbb{Q}(\alpha)$. Let $\mathbb{C} \supseteq L \supseteq \mathbb{Q}$ be the Galois closure of $K/\mathbb{Q}$ seen as a subfield of $\mathbb{C}$.

(i) Show that $[L : \mathbb{Q}] = 6$, determine the Galois group of $L/\mathbb{Q}$ and show that $F = \mathbb{Q}(\sqrt{-3}) \subseteq L$.
(ii) Determine the factorization of the primes $p = 2, 3, 5$ in $L/\mathbb{Q}$.
(iii) Determine the set of ramified primes in $L/\mathbb{Q}$.

Exercise 2. Let K be a number field. Remember that we proved in exercise 4 of Exercise sheet 5 that given an ideal I of $\mathcal{O}_K$ there exists a finite extension extension L of K such that $I\mathcal{O}_L$ is principal.

(i) Is there a finite extension F of K such that for every ideal $I \subset \mathcal{O}_K$ the ideal $I\mathcal{O}_F$ is principal?
(ii) For $K = \mathbb{Q}(\sqrt{-30})$, can you construct explicitly an extension F/K as in (i)(Use exercise 2 of exercise sheet 7)?
(iii) Is F of point (ii) unique?

Exercise 3. Let $d > 2$ be a squarefree positive integer and q be a prime not dividing d. Assume $q \equiv 3 \pmod 4$. Let $F = \mathbb{Q}(\sqrt{-dq})$ and $E = F(\sqrt{-q}) = \mathbb{Q}(\sqrt{-dq}, \sqrt{-q})$. The goal of this exercise is to show that $\mathcal{O}_E$ is not free as an $\mathcal{O}_F$-module.

(i) Suppose that $\mathcal{O}_E$ is free as an $\mathcal{O}_F$-module: then, prove that $\{1, (1 + \sqrt{-q})/2\}$ is a basis of $\mathcal{O}_E$ over $\mathcal{O}_F$. (Hint: Take a basis $\{e_1, e_2\}$ of $\mathcal{O}_E$ over $\mathcal{O}_F$, write $\{1, (1+\sqrt{-q})/2\}$ as linear combination of e_1 and e_2, prove, using the fact that E/F is Galois, that the matrix given by the coefficients of e_1, e_2 in the prevoius expressions has determinant in $\mathcal{O}_F^\times$)
(ii) Prove that $\{1, (1 + \sqrt{-q})/2\}$ is NOT a basis of $\mathcal{O}_E$ over $\mathcal{O}_F$. (Hint: Suppose that it is a basis and try to write $\sqrt{d} = \frac{\sqrt{-dq}}{\sqrt{-q}}$ in terms of this basis).

Exercise 4.
Let K be a number field whose class group Cl_K has order h, and let L/K be an extension of degree $[L : K] = n$. Assuming that $(h, n) = 1$

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on June 17^{th}.

prove that the extension map $\mathrm{Cl}_K \to \mathrm{Cl}_L$ given in exercise 4 in exercise sheet 7 is injective.