June 1st, 2016

Number Theory II

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Exercise sheet 7[1]

Exercise 1. Use the exercise 2 of exercise sheet 6 to understand ramification in quadratic fields.

Let $m \neq 1$ be a square free integer and $K = \mathbb{Q}(\sqrt{m})$. Characterize the ramified, the inert and the split primes.

Exercise 2. Consider the number field $K = \mathbb{Q}(\sqrt{-30})$. Show that the class group of $\mathcal{O}_K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Exercise 3. Is there a totally ramified prime in the extension

$$\mathbb{Q}(\sqrt{-d}, \sqrt{d})/\mathbb{Q}$$

where $d > 0$ is odd and square-free?

Exercise 4. Let L/K be a finite Galois extension of number fields and set $G = \mathrm{Gal}(L/K)$.

(i) Prove that the action of G on $\mathcal{O}_L$ defines an action on the fractional ideals Id_L and also an action on the ideal class group Cl_L.

(ii) Show that the extension map $\varphi\colon \mathrm{Id}_K \to \mathrm{Id}_L$ given by $I \mapsto I\mathcal{O}_L$ induces a map $\varphi\colon \mathrm{Cl}_K \to \mathrm{Cl}_L$.

(iii) Determine a necessary and sufficient condition on L/K to ensure that the "Galois principle holds for ideals", namely that

$$\{I \in \mathrm{Id}_L \text{ such that } \sigma(I) = I \;\forall\, \sigma \in G\} = \mathrm{Id}_K$$

where Id_K is seen as a subgroup of Id_L through the extension map $I \mapsto I\mathcal{O}_L$.

(iv) Provide a counterexample to the "Galois principle for ideal classes", so find a finite Galois extension L/K of number fields such that

$$\{x \in \mathrm{Cl}_L \text{ such that } \sigma(x) = x\} \neq \varphi(\mathrm{Cl}_K)$$

in the notations of point (ii).

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on June 10th.