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Number Theory II

Prof. H. Esnault, Dr. V. Di Proietto

Exercise sheet 2[1]

Exercise 1. Let $A = \mathbb{Z}$ and let M be a free $\mathbb{Z}$-module of rank n. Let $N \subseteq M$ be a submodule.

(i) Show that N is a free $\mathbb{Z}$-module of rank $\leq n$

(ii) Show that the commutative group M/N is finite if and only if the rank of N is n.

(iii) Let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis of M, and let $f_1, \ldots, f_n$ be a basis of N, then there exist $a_{i,j} \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ such that $f_i = \sum_j a_{i,j} e_j$. Show that $[M : N] = |\det(a_{ij})_{i,j}|$.

Exercise 2. Let K be a number field and let $\mathcal{O}_K$ be the ring of integers of K. Suppose that $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module (you will see that this is always the case). Prove that, if $\alpha \in \mathcal{O}_K$, then the absolute value of the norm $\mathrm{Nm}(\alpha)$ coincides with the order of the group $\mathcal{O}_K/\alpha\mathcal{O}_K$.

Exercise 3. Let A be an integrally closed ring, and let K be its field of fractions. Let $f(X) \in A[X]$ be a monic polynomial. If $f(X)$ is reducible in $K[X]$, then it is reducible in $A[X]$.

Exercise 4. Let $K \subset L$ be an extension of fields. Show that if L/K is not separable, then $\mathrm{disc}(L/K) = 0$.

Exercise 5. Remember from **Ex.1** of Exercise sheet 1 that the ring of integers $\mathcal{O}_K$ of $K = \sqrt{d}$, where d is a square-free integer, is a free $\mathbb{Z}$-module of rank 2.

(i) Using the basis of $\mathcal{O}_K$ we found in **Ex.1** of Exercise sheet 1, calculate $\mathrm{disc}(\mathcal{O}_K/\mathbb{Z})$.

(ii) Calculate $[\mathcal{O}_K : \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}]$.

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on May 6th.