ZAHLENTHEORIE II – ÜBUNGSBLATT 8

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Exercise 1. Let K be a field, and let $G \subseteq K^*$ be a finite subgroup with respect to the multiplicative structure of K. The goal of this exercise is to show that G is an acyclic group.

(a) Show that in a finite group any element has a finite order and the orders are bounded by the order of the finite group.
(b) Show that if $a \in G$ has order $s \in \mathbb{N}^+$ and $b \in G$ has order $r \in \mathbb{N}^+$, and if s, r are coprime to each other, then ab has order sr.
(c) Let $a_0 \in G$ be an element in G which has the maximal order, say $n \in \mathbb{N}^+$. Show that if $b \in G$ is any element with order m, then $m|n$.
(d) Show that G is cyclic whose generator can be chosen to be a_0.
(e) Where did we use the assumption that K is a field?

Exercise 2. Let A be a Dedekind domain, K be its fraction field, and let I denote the multiplicative group of non-zero fractional ideals. Let $P \subseteq I$ be the subset principal ideals, i.e. P be the image of the map $K^* \to I$ sending $a \in K^* := K \setminus \{0\}$ to the fractional ideal generated by a. Show that P is a subgroup of I and that we have an exact sequence of groups

$$1 \to U \to K^* \to I \to H \to 1$$

where U is the group of invertible elements in A, $H := I/P$.

Exercise 3. If $[K : \mathbb{Q}]$ is odd, show that $\boldsymbol{\mu}(K) = \{\pm 1\}$, i.e. the only roots of unity in K are ± 1. (Hint: Show that in this case there is a real embedding.)

Exercise 4. (a) If $K := \mathbb{Q}(\sqrt{-1})$, what is the group of units of $\mathcal{O}_K$?
(b) If $K := \mathbb{Q}(\sqrt{2})$, what is the group of units of $\mathcal{O}_K$?

If you want your solutions to be corrected, please hand them in just before the lecture on June 13, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.