ZAHLENTHEORIE II – ÜBUNGSBLATT 7

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Exercise 1. Let A be a Dedekind domain with fraction field K, and let L/K be a finite separable extension. Let B be the integral closure of A in L. In the class we have shown that if B is free over A, then a maximal ideal $\mathfrak{p}$ in A divides $\operatorname{disc}(B/A)$ if and only if $\mathfrak{p}$ is ramified. In this exercise we show that this characterization of discriminant leads to a new definition of $\operatorname{disc}(B/A)$ without the assumption that B is free over A.

(a) Let $\mathfrak{p}$ be a maximal ideal of A, and let $S := A \setminus \mathfrak{p}$. Then $S^{-1}B$ is free over $S^{-1}A$ so we can define $\operatorname{disc}(S^{-1}B/S^{-1}A)$. Show that $\operatorname{disc}(S^{-1}B/S^{-1}A) = \mathfrak{p}^{m_\mathfrak{p}}(S^{-1}A))$ for some $m_\mathfrak{p} \in \mathbb{N}$.

(b) Show that there exists a non-zero element $f \in A$ such that B_f is free over A_f.

(c) Let $\mathfrak{p}$ be a maximal ideal of A, and let $S := A \setminus \mathfrak{p}$. Show that $\operatorname{disc}(S^{-1}B/S^{-1}A) = S^{-1}\operatorname{disc}(B_f/A_f)$.

(d) Show that $m_\mathfrak{p} = 0$ for all but finitely many $\mathfrak{p}$ in A.

(e) Now define

$$\operatorname{disc}(B/A) := \prod_{\mathfrak{p} \text{ maximal ideal in } A} \mathfrak{p}^{m_\mathfrak{p}}$$

Show that $\mathfrak{p} | \operatorname{disc}(B/A)$ if and only if $\mathfrak{p}$ is ramified in B.

(f) Show that if B is free over A then the discriminant we just defined coincides with the one we defined in the class.

Exercise 2. Let K be a number field, and let $n := [K : \mathbb{Q}]$. Let $\mathfrak{a} \subseteq \mathcal{O}_K$ be a non-zero ideal.

(a) Show that $\mathfrak{a}$ is a free $\mathbb{Z}$ module.

(b) Show that if $S := \mathbb{Z} \setminus \{0\}$, then $S^{-1}\mathcal{O}_K = K$.

(c) Show that if $S := \mathbb{Z} \setminus \{0\}$, then $S^{-1}\mathfrak{a}$ is an n-dimensional $\mathbb{Q}$-vector space.

(d) Show that $\mathfrak{a}$ has rank n over $\mathbb{Z}$.

If you want your solutions to be corrected, please hand them in just before the lecture on June 6, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.