ZAHLENTHEORIE II – ÜBUNGSBLATT 2

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Exercise 1. In this exercise we recall some basic properties of Noetherian rings and Noetherian modules. Let A be a commutative ring, and let M be an A-module. Recall that the module M is called Noetherian if any assending chain $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$ of submodules of M ends, i.e. there exists some $n \in \mathbb{N}$ such that $M_n = M$. This is clearly equivalent to the condition that any submodule of M is finitely generated. The ring A is called a Noetherian ring if it is a Noetherian module viewed as a rank 1 free module over itself.

(a) Show that if
$$0 \to M' \to M \to M'' \to 0$$
is an exact sequence of A-modules, then M is Noetherian if and only if M' and M'' both are Noetherian.

(b) Show that a finite direct sum of Noetherian modules is still Noetherian.

(c) Show that if A is a Noetherian ring, then any finitely generated A-module is Noetherian.

(d) Show that if A is a Noetherian ring and if M is a finitely generated A-module, then any submodule of M is Noetherian.

Exercise 2. In this exercise we recall some basic properties of finitely generated free modules.. Let A be a non-zero commutative ring.

(a) Show that $A^{\bigoplus m} \cong A^{\bigoplus n}$ for $n \in \mathbb{N}$ and $m \in \mathbb{N}$ if and only if $m = n$. In this way for any finitely generated torsion free A-module M we can define its rank to be the unique natural number n such that $M \cong A^{\bigoplus n}$.

(b) Show that if A is an integral domain, and if $A^{\bigoplus m} \hookrightarrow A^{\bigoplus n}$ is injective with $m, n \in \mathbb{N}$, then $m \leq n$.

(c) Let A be a principal ideal domain. Show that a finitely generated A-module is free if and only if it is torsion free.

(d) Let A be a principal ideal domain. Show that if M is a finitely generated torsion free A-module and $N \subseteq M$ is a submodule, then the rank of N is less or equal to the rank of M.

Remark. In fact more is true. If A is a non-zero commutative ring, and if $A^{\bigoplus m} \hookrightarrow A^{\bigoplus n}$ is injective, then $m \leq n$. If not you can set $\phi : A^{\bigoplus m} \hookrightarrow A^{\bigoplus n} \subseteq A^{\bigoplus m}$, where the last embedding is the map sending coordinates in $A^{\bigoplus n}$ to the first n-coordinates in $A^{\bigoplus m}$. Then we can use Cayley-Hamilton theorem to find a minimal non-zero monic polynomial
$$a_0 + a_1 X + a_2 X^2 + \cdots + a_{n-1} X^{n-1} + a_n X^n$$
which kills ϕ. Using the injectivity of ϕ one sees immediately that $a_0 \neq 0$. But then there would be a contradiction because the value of $(0, \cdots, 0, 1)$ under
$$a_0 + a_1 \phi + a_2 \phi^2 + \cdots + a_{n-1} \phi^{n-1} + a_n \phi^n$$

If you want your solutions to be corrected, please hand them in just before the lecture on May 2, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.

can not be 0.

Exercise 3. Let L/K be a finite field extension. Let A be an integrally closed domain whose fraction field is K, and let B be the integral closure of A in L.

(a) Let $e_1, e_2, \cdots, e_n$ be a basis of L/K and suppose that $e_i \in B$ for all $1 \leq i \leq n$. Let $C\colon = \sum_{1 \leq i \leq n} Ae_i \subseteq B$ Define
$$C^*\colon = \{\alpha \in L | \operatorname{Tr}(\alpha\gamma) \in A,\ \forall\, \gamma \in C\}$$
Show that $\alpha \in C^*$ if and only if $\operatorname{Tr}(\alpha e_i) \in A$ for all i, and therefore we have
$$C \subseteq B \subseteq C^*$$
In the course we defined C^* with the help of dual basis, but if L/K is not separable, then the bilinear form $\operatorname{Tr}(-,-)$ is not non-degenerate (cf. Ex 1.1 e)), so there is no dual basis in this case. What we have just defined can be seen as a generalization of C^* to arbitrary finite field extension:

(b) Notations being as above, assume further that L/K is separable, and let $\{e'_i\}_{1 \leq i \leq n}$ be the dual basis of $\{e_i\}_{1 \leq i \leq n}$. Show that
$$C^* = \sum_{1 \leq i \leq n} Ae'_i$$

(c) Now suppose that $L = K(\alpha)$ and that L/K is separable. Let $f(X)$ be the minimal polynomial of α over K, and let
$$\frac{f(X)}{(X-\alpha)}\colon = \beta_0 + \beta_1 X + \cdots + \beta_{n-1}X^{n-1}$$
Let $\alpha_1, \cdots, \alpha_n$ be the roots of $f(X)$ in the algebraic closure of K. Show that
$$\sum_{i=1}^{n} \frac{f(X)}{(X-\alpha_i)} \frac{\alpha_i^r}{f'(X)} = X^r$$
where $f'(X)$ is the formal derivative of $f(X)$.

(d) Show that if we define the trace of a polynomial with coefficients in L to be the polynomial obtained by applying the trace to the coefficients, then
$$\operatorname{Tr}[\frac{f(X)}{(X-\alpha)} \frac{\alpha^r}{f'(X)}] = X^r$$

(e) Conclude that $\{\frac{\beta_j}{f'(\alpha)}\}_{1 \leq j \leq n}$ is the dual basis of $\{\alpha^i\}_{1 \leq i \leq n}$. Therefore if $C = \sum_{i=1}^{n} A\alpha^i$ then $C^* = \sum_{j=1}^{n} A\frac{\beta_j}{f'(\alpha)}$.