ZAHLENTHEORIE II – ÜBUNGSBLATT 2

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let A be a Dedekind domain, $K = \mathrm{Frac}(A)$, L/K a finite separable extension and B the integral closure of A in L. Assume that there exists some $\beta \in B$ such that $B = A[\beta]$. Let $f(T) \in K[T]$ be the monic minimal polynomial of β.

(a) Show that $f(T) \in A[T]$.

(b) Show that $A[T]/(f(T)) = A[\beta]$.

(c) If $\mathfrak{p} \subseteq A$ is a prime ideal, show that there exist monic polynomials $h_1(T), \dots, h_g(T) \in A[T]$, and $e_1, \dots, e_g \in \mathbb{N}_{>0}$ such that

$$f(T) \equiv h_1(T)^{e_1} \cdot \ldots \cdot h_g(T)^{e_g} \mod \mathfrak{p},$$

and such that the reductions $\bar{h}_i(T) \in (A/\mathfrak{p})[T]$ are pairwise distinct and irreducible.

(d) Show that the distinct prime ideals of B lying over $\mathfrak{p}$ are given by

$$\mathfrak{P}_i := (\mathfrak{p}, h_i(\beta)), \quad i = 1, \dots, g.$$

(e) Show that

$$\mathfrak{p}B = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i},$$

and $[B/\mathfrak{P}_i : A/\mathfrak{p}] = \deg h_i$.

Exercise 2. Let A be a commutative ring, and let $S \subseteq A$ be a multiplicative subset of A. Let M be an A-module.

(a) Show that the localization map $\lambda\colon M \to S^{-1}M$ has the following universal property: If N is an $S^{-1}A$-module, and if $\phi\colon M \to N$ is an A-linear map, then there exists a unique map of $S^{-1}A$-modules $\psi\colon S^{-1}M \to N$ such that $\psi \circ \lambda = \phi$.

(b) Suppose that $\mathfrak{m}$ is a maximal ideal of A, and $S = A \setminus \mathfrak{m}$. Then we have

$$(M/\mathfrak{m}M)_{\mathfrak{m}} = M/\mathfrak{m}M$$

as $S^{-1}A$-modules.

If you want your solutions to be corrected, please hand them in just before the lecture on May 08, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.