ZAHLENTHEORIE II – ÜBUNGSBLATT 1

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. (a) Let K be a field. Show that the polynomial ring $K[X, Y]$ is not Dedekind.
(b) Show that $\mathbb{Z}[\sqrt{5}]$ is not Dedekind. (Hint: Show that $\frac{1+\sqrt{5}}{2} \notin \mathbb{Z}[\sqrt{5}]$.)

Exercise 2. Let A be an integral domain, and let K be its fraction field. We call a sub A-module $M \subseteq K$ an *invertible ideal* if there is a sub A-module $N \subseteq K$ such that $MN = A$.

(a) Show that if such N exists then it is equal to the following
$$(A : M) = \{x \in K | xm \in A \; \forall \; m \in M\}$$
(b) Show that if $M \subseteq K$ is invertible then it is finitely generated.
(c) Let $M \subseteq K$ be a fractional ideal. Show that the following are equivalent:
  (1) M is invertible.
  (2) M is finitely generated, and for all prime ideal $\mathfrak{p} \subseteq A$ the submodule $M_{\mathfrak{p}}$ is invertible.
  (3) M is finitely generated, and for all maximal ideal $\mathfrak{m} \subseteq A$ the submodule $M_{\mathfrak{m}}$ is invertible.

  (Hint: We have to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Here one has to show $(A_{\mathfrak{m}} : M_{\mathfrak{m}}) = (A : M)_{\mathfrak{m}}$ under the assumption that M is finitely generated.)

Exercise 3. Let A be a local integral domain with a maximal ideal $\mathfrak{m} \neq 0$.

(a) Show that if A is a DVR, then any fractional ideal of A is invertible. (Hint: Every non-zero ideal of A is of the form (π^i) for a fixed uniformizer π and a natural number i.)
(b) Show that if every fractional ideal of A is invertible then A is Noetherian.
(c) Show that if every fractional ideal of A is invertible then any non-zero ideal $\mathfrak{a}$ is a power of $\mathfrak{m}$, i.e. $\mathfrak{a} = \mathfrak{m}^i$ for some $i \in \mathbb{N}$. (Hint: Otherwise the set of ideals which are not of the form $\mathfrak{m}^i$ is non-empty, and suppose $\mathfrak{a}$ is a maximal element in this set. Show that we have the inequalities $\mathfrak{a} \subseteq \mathfrak{m}^{-1}\mathfrak{a} \subseteq A$. Then $\mathfrak{m}^{-1}\mathfrak{a}$ is of the form $\mathfrak{m}^i$.)
(d) Show that if every fractional ideal of A is invertible then A is a DVR. (Hint: The ideal $\mathfrak{m}^i$ is a prime ideal if and only if $i = 1$.)

Exercise 4. Let A be an integral domain.

(a) Suppose that A is Dedekind. Use Ex.2 and Ex.3 to show that every fractional ideal is invertible.
(b) Suppose that every fractional ideal in A is invertible. Show that A is Noetherian. (Hint: Use Ex.2 (b).)
(c) Suppose that every fractional ideal in A is invertible. Show that A is Dedekind. (Hint: Use Ex.3 (d) and what we did in the lecture.)

If you want your solutions to be corrected, please hand them in just before the lecture on April 24, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.