ZAHLENTHEORIE II – ÜBUNGSBLATT 11

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let R be a Dedekind domain with field of fractions K.

(a) Given a non-zero prime ideal $\mathfrak{p} \subseteq R$, define $v_{\mathfrak{p}} : K \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $v_{\mathfrak{p}}(0) = \infty$, $v_{\mathfrak{p}}(x) = \max\{n | x \in \mathfrak{p}^n\}$ for $a \in R$ and

$$v_{\mathfrak{p}}\left(\frac{x}{y}\right) = v_{\mathfrak{p}}(x) - v_{\mathfrak{p}}(y)$$

for $x, y \in K$.

Show that $v_{\mathfrak{p}}$ defines a non-archimedean discrete absolute value $|\cdot|_{\mathfrak{p}} = \exp(-v_{\mathfrak{p}}(\cdot))$ on K with valuation ring

$$R_{\mathfrak{p}} := \left\{ \frac{x}{y} \in K \,\middle|\, y \notin \mathfrak{p} \right\}.$$

(b) Assume that $|\cdot|$ is a non-trivial non-archimedean absolute value on K. Let $\mathcal{O}_K \subseteq K$ be the associated valuation ring. Show that if $R \subseteq \mathcal{O}_K$, then $|\cdot|$ is equivalent to $|\cdot|_{\mathfrak{p}}$ for some nonzero prime $\mathfrak{p} \subseteq R$. (*Hint*: You can imitate the part of the proof of Ostrowski's theorem classifying non-archimedean absolute values on $\mathbb{Q}$, where $\mathbb{Z} \subseteq A$ is replaced by $R \subseteq O_K$.)

Exercise 2. Let $p \in \mathbb{Z}$ be prime and $|\cdot|$ an absolute value on the field of rational functions $\mathbb{F}_p(T)$ which is not trivial. Prove that $|\cdot|$ is equivalent to one of the following absolute values:

(a) For $f(T) \in \mathbb{F}_p(T)$ irreducible, define for $g(T) \in \mathbb{F}_p(T)$

$$|g(T)|_f := \exp(-\max\{n | g(T) \in (f(T))^n\})$$

and for $g(T)/h(T) \in \mathbb{F}_p(T)$:

$$\left|\frac{g(T)}{h(T)}\right|_f := \frac{|g(T)|_f}{|h(T)|_f}.$$

(b) For $\frac{g(T)}{h(T)} \in \mathbb{F}_p(T)$:

$$\left|\frac{g(T)}{h(T)}\right|_\infty := 2^{\deg(g(T)) - \deg(h(T))}.$$

(*Hint: Distinguish the cases $|T| \leq 1$ and $|T| > 1$ and use the previous exercise for the first case.* For the second case first show that for any $g(T) \in \mathbb{F}_p[T]$ we have $|g(T)| \leq |T|^{\deg(g(T))}$. Suppose that the leading coefficient of $g(T)$ is a_0. Then observe that $|g(T)| \geq |T|^{\deg(g(T))} - |g(T) - a_0 T^{\deg(g(T))}| \geq |T|^{\deg(g(T))} - |T|^{\deg(g(T))-1} = |T|^{\deg(g(T))}(1 - \frac{1}{|T|})$.)

If you want your solutions to be corrected, please hand them in just before the lecture on Juli 4, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.