ZAHLENTHEORIE II – ÜBUNGSBLATT 7

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Exercise 1. If $[K:\mathbb{Q}]$ is odd, show that $\boldsymbol{\mu}(K)=\{\pm 1\}$, i.e. the only roots of unity in K are ± 1. (Hint: Show that in this case there is a real embedding.)

Exercise 2. (a) If $K:=\mathbb{Q}(\sqrt{-1})$, what is the group of units of $\mathcal{O}_K$?
(b) If $K:=\mathbb{Q}(\sqrt{2})$, what is the group of units of $\mathcal{O}_K$?

Exercise 3. Let K be a number field. Recall that K is called *totally real* if all its embeddings into $\mathbb{C}$ lie in $\mathbb{R}$, and K is called *totally imaginary* if all its embeddings into $\mathbb{C}$ do not lie in $\mathbb{R}$. The field K is called a *CM field* if K is a quadratic extension of a totally real number field F.

(a) Show that a number field K is a CM field if and only if there exists a totally real field F and an element $\alpha\in F$ such that $K=F(\sqrt{\alpha})$, where α is an element in F the image of which under any embedding $F\to\mathbb{C}$ is a negative real number.
(b) Let K be a CM field. Compute the rank of the unit group U_K of K.
(c) Let F be a totally real field, and let K be a totally imaginary quadratic extension of F. Compute the rank of the unit group U_F of F, and conclude that $U_F\subseteq U_K$ is a subgroup of finite index.
(d) Let F be a totally real field, and let K be a totally imaginary quadratic extension of F. Let $a\mapsto\bar{a}$ be the nontrivial automorphism of K fixing F. Then $\rho(\bar{a})=\overline{\rho(a)}$ for all homomorphisms $\rho:K\to\mathbb{C}$, where $\overline{\rho(a)}$ means the complex conjugate of the complex number $\rho(a)$.
(e) Notations and conventions are as in (d). Show that for all $a\in U_K$ the element $\frac{a}{\bar{a}}\in\mu(K)$. (Hint: We have shown in the class that an algebraic integer is in $\mu(K)$ if all of its conjugates in $\mathbb{C}$ have absolute value 1.)
(f) Notations and conventions are as in (e). Show that the map

$$\phi: U_K\to\mu(K)/\mu(K)^2$$
$$a\mapsto\frac{a}{\bar{a}}$$

is a group homomorphism.
(g) Notations and conventions are as in (f). Show that $\operatorname{Ker}(\phi)=\mu(K)U_F$.
(h) Notations and conventions are as in (g). Show that $\mu(K)U_F$ as a subgroup of U_K has index 1 or 2.

Exercise 4. Let ζ_n be a primitive n-th root of 1, $n>2$. Define $K:=\mathbb{Q}(\zeta_n)$ and $F:=\mathbb{Q}(\zeta_n+\zeta_n^{-1})$.

If you want your solutions to be corrected, please hand them in just before the lecture on Juni 19, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.

(a) Show that F is totally real.
(b) Show that K is totally imaginary.
(c) Show that K is a CM field.

Exercise 5. Let ζ_n be a primitive n-th root of 1. Let $K \coloneqq \mathbb{Q}(\zeta_n)$.

(a) Show that all embeddings from $K \to \bar{\mathbb{Q}}$ sends ζ_n to an n-th primitive root of 1.
(b) Conclude that $K/\mathbb{Q}$ is Galois.
(c) For any prime $p \nmid n$ set the minimal polynomials of ζ_n and ζ_n^p to be $f(T)$ and $g(T)$ respectively. Show that they are nomic polynomials in $\mathbb{Z}[T]$ dividing $T^n - 1$.
(d) Let $\bar{f}[T]$ and $\bar{g}[T]$ be the reductions of $f(T)$ and $g(T)$ mod p respectively. Show that if $f(T) \neq g(T)$ then $\bar{f}[T]$ and $\bar{g}[T]$ are relatively coprime in $\mathbb{F}_p[T]$.
(e) Show that every prime factor of $\bar{f}[T]$ divides $\bar{g}[T]$.
(f) Conclude that $f(T) = g(T)$.
(g) Conclude that $\mathrm{Gal}(K/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^*$.