ZAHLENTHEORIE II – ÜBUNGSBLATT 6

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Exercise 1. Let K be a number field. In this exercise we prove that the class group $\mathrm{Cl}(K)$ of K is finite. A different proof has been presented in the lectures.

Let $x_1, \dots, x_n$ be an integral basis of K and $\sigma_1, \dots, \sigma_n : K \hookrightarrow \mathbb{C}$ the embeddings of K into $\mathbb{C}$. Define

$$M_K := \prod_{j=1}^{n} \sum_{i=1}^{n} |\sigma_j(x_i)| \in \mathbb{R}_{>0}.$$

(a) Using the methods from the lecture, show that $\mathrm{Cl}(K)$ is finite if and only if there exists a number $M \in \mathbb{R}_{>0}$ such that for every nonzero ideal $I \subseteq \mathcal{O}_K$ there exists $a \in I \setminus \{0\}$ such that

$$|\operatorname{Nm}_{K/\mathbb{Q}}(a)| \leq M \cdot \mathbb{N}(I).$$

We will see that M_K has this property and hence that $\mathrm{Cl}(K)$ is finite.

(b) Given a nonzero ideal $I \subseteq \mathcal{O}_K$, let $k \in \mathbb{N}$ such that $k^n \leq \mathbb{N}(I) < (k+1)^n$. Use the "Pigeonhole principle" to show that there exists an element $a := \sum_{i=1}^n a_i x_i \in I \setminus \{0\}$ such that $|a_i| \leq k$ for $i = 1, \dots, n$. (Hint: Consider the set $S := \{\alpha = \sum_{i=1}^n a_i x_i | 0 \leq a_i \leq k\}$ in $\mathcal{O}_K/I$.)

(c) Conclude that

$$|\operatorname{Nm}_{K/\mathbb{Q}}(a)| \leq \mathbb{N}(I) \cdot M_K$$

and hence that $\mathrm{Cl}(K)$ is finite.

Exercise 2. Let $n \in \mathbb{N}$ and $\Lambda \subseteq \mathbb{R}^n$ a full lattice and D a fundamental parallelepiped for Λ. Let μ denote the Lebesgue measure on $\mathbb{R}^n$.

(a) Find a set $S \subseteq \mathbb{R}^n$ which is convex, symmetric around the origin and such that $\mu(S) = 2^n \mu(D)$, but $S \cap \Lambda = \{0\}$.

(b) Why is this not a contradiction to Minkowski's theorem?

Exercise 3. Let K be a number field, and let $\alpha \in K$ be an element.

(a) Show that α is a unit iff $\alpha \in \mathcal{O}_K$ and $\operatorname{Nm}_{K/\mathbb{Q}}(\alpha) = \pm 1$.

(b) Let d be a square free integer. Show that

$$\begin{cases} m + n\sqrt{d} \text{ is a unit iff } m^2 - n^2 d = \pm 1, & d \equiv 2,3 \mod 4 \\ m + n\frac{1+\sqrt{d}}{2} \text{ is a unit iff } (2m+n)^2 - dn^2 = \pm 4, & d \equiv 1 \mod 4 \end{cases}$$

(c) Suppose that $d < 0$. List all the possible d so that $\mu(\mathbb{Q}(\sqrt{d})) \neq \{\pm 1\}$.

Exercise 4. Show that if K is a field, then any finite subgroup of $K^\times$ is cyclic. We have used this in the class. Now its your turn to prove it.

If you want your solutions to be corrected, please hand them in just before the lecture on Juni 5, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.

Exercise 5. (Extra) Let K be a number field. We define $\widetilde{\mu}(K) \subseteq K^\times$ as the subgroup of elements $\alpha \in K^\times$ such that, for every conjugate α' of α, the absolute value of α' is 1. In class we have proven that $\widetilde{\mu}(K) \cap \mathcal{O}_K = \mu(K)$.

(a) Suppose that K admits a real embedding, show that $\widetilde{\mu}(K) = \mu(K) = \{\pm 1\}$.
(b) Deduce that if $n > 2$, the cyclotomic field $\mathbb{Q}(\zeta_n)$ is totally imaginary.
(c) Find a number field K such that $\widetilde{\mu}(K) \neq \mu(K)$.