ZAHLENTHEORIE II – ÜBUNGSBLATT 5

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let $K = \mathbb{Q}[\sqrt{-5}]$.

(a) Compute the discriminant of $K/\mathbb{Q}$.
(b) Show that $\mathbb{Z}[\sqrt{-5}]$ is the ring of integers of K.
(c) Compute the group $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}])$.

Exercise 2. Let K be a number field, and let $I \subseteq \mathcal{O}_K$ be an ideal.

(a) Use the fact that $\mathrm{Cl}(\mathcal{O}_K)$ is finite to show that there is a finite extension L/K such that $I\mathcal{O}_L$ is a principal ideal. (Hint: There exists $n \in \mathbb{N}$ such that $I^n = (a)$ for some $a \in \mathcal{O}_K$. Now consider $a^{\frac{1}{n}}$. Show that $I = (a^{\frac{1}{n}})$.)
(b) Show that there is a finite extension L/K such that for any ideal $J \subseteq \mathcal{O}_K$ the extension $J\mathcal{O}_L$ is a principal ideal.

Exercise 3. Let K be a number field with ring of integers $\mathcal{O}_K$.

(a) If $I \subseteq J \subseteq K$ two fractional ideals of $\mathcal{O}_K$. Show that $IJ^{-1} \subseteq \mathcal{O}_K$ and that
$$[J : I] = \mathbb{N}(IJ^{-1}).$$
Here $[J : I]$ is the index of I in J, i.e. the cardinatlity of the abelian group J/I.
(b) If $I \subseteq \mathcal{O}_K$ is a nonzero ideal, show that I is principal if and only if there is $a \in \mathcal{O}_K$ such that
$$|N_{K/\mathbb{Q}}(a)| = \mathbb{N}(I).$$

If you want your solutions to be corrected, please hand them in just before the lecture on Juni 2, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.