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Algebraic Groups

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Exercise sheet 8[1]

Exercise 1. Let G be a group scheme over a field k, and let k'/k be a purely separable field extension. Show that if $G \times_k k'$ is diagonalizable then G is also diagonalizable. (Hint: Use what we have proved in the leture: If $G = \mathrm{Spec}\,(A)$, then G is diagonalizable iff for any sub vector space $U \subseteq A$ satisfying $\Delta(U) \subseteq U \otimes_k U$, $U^{\checkmark}$ is equal to $\prod_{1\leq i\leq n} k$. Note that since k'/k is purely inseparable, if K/k is a finite separable extension, then $K \otimes_k k'$ is a field, so if $K \otimes_k k' = \prod_{1\leq i\leq n} k'$ then $n = 1$ and $K = k$.)

Exercise 2. Let

$$1 \to G' \to G \to G'' \to 1$$

be an exact sequence of affine k-group schemes with G', G'' being of multiplicative type. Show that G is of multiplicative type if and only if it is commutative. (Hint: Use the fact that G is multiplicative iff it is commutative and $\mathrm{Hom}_{k-\mathrm{grp.sch}}(G, \mathbb{G}_a) = 0$.)

Exercise 3. Let G be a trigonalizable group scheme over k. Show that there exists a normal series of G:

$$G \supsetneq G_0 \supseteq G_1 \cdots G_r = \{0\}$$

with $G^u = G_0$ such that G_i/G_{i+1} is either $\mathbb{G}_a$ or α_{p^n} or a finite étale group scheme over k. Moreover if $k = \bar{k}$ is algebraically closed then you can choose the normal series so that G_i/G_{i+1} is either $\mathbb{G}_a$ or α_{p^n} or $(\mathbb{Z}/p\mathbb{Z})_k$.

Exercise 4. Show that an affine commutative group scheme over an algebraically closed field is always trigonalizable.

Exercise 5. In the class we defined $\mathbb{U}_{n,k}$ (resp. $\mathbb{T}_{n,k}$) to be the subgroup of $\mathrm{GL}_{n,k}$ of upper triangular matrices with 1 in the diagonals (resp. upper triangular matrices). Check that we have an exact sequence

$$1 \to \mathbb{U}_{n,k} \to \mathbb{T}_{n,k} \to \mathbb{G}_m^n \to 1$$

[1]If you want your solutions to be corrected, please hand them in just before the lecture on June 15, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

and use it to prove that $\mathbb{T}_{n,k}$ is neither unipotent nor multiplicative when $n > 1$.

Exercise 6. Let H, N be two group schemes over a scheme S. If H acts on N via group automorphisms, i.e. there is a morphism of sheaves of groups $\phi : H \to \mathrm{Aut}_{k-\mathrm{grp.sch}}(N)$, then we can define a new group $N \rtimes_\phi H$, *the semi-direct product of N and H along ϕ*, whose underlying space is $N \times_S H$, whose multiplication is defined by $(n_1, h_1) \cdot (n_2, h_2) := (n_1\phi(h_1)(n_2), h_1h_2)$. Show that there exists an exact sequence

$$1 \to N \to N \rtimes_\phi H \to H \to 1$$

and there is a splitting $H \to N \rtimes_\phi H$ given by $h \mapsto (e, h)$. Conversely, given an exact sequence

$$1 \to N \to G \to H \to 1$$

of S-group schemes and a splitting $r : H \to G$, we get a action $\phi : H \to \mathrm{Aut}_{k-\mathrm{grp.sch}}(N)N$ given by $h \mapsto h(-)h^{-1}$. Show that there is a canonical isomorphism $G \to N \rtimes_\phi H$.

Using the tool of semi-direct product we can construct a lot of new groups starting with some known group schemes. For example, $\mathbb{G}_m$ acts on $\mathbb{G}_a$ via the scalar multiplication. So we can define a semi-direct product $\mathbb{G}_a \rtimes \mathbb{G}_m$ using this action. Show that $\mathbb{G}_a \rtimes \mathbb{G}_m$ is trigonalizable but neither multiplicative nor unipotent.