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# Algebraic Groups

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## Exercise sheet 7[1]

**Exercise 1.** Let $G$ be an affine group scheme over $k$.

(1) Show that $G$ is finite over $k$ if and only if $G \times_k \bar{k}$ is finite over $\bar{k}$;
(2) Show that if $k = \bar{k}$, and if $G \to G'$ is a map of affine group schemes in which $G'$ is finite, then $G$ is finite if the kernel is finite.
(3) Show that the condition *finiteness* on affine group schemes is stable under extensions.

**Exercise 2.** Let $G$ be an affine group scheme over $k$. Show that $G$ is unipotent if and only if there is a central normal series

$$G = G_0 \supsetneq G_1 \supseteq G_2 \cdots$$

such that each quotient $G_i/G_{i+1}$ is either $\mathbb{G}_a$ or $\alpha_{p^n}$ or a finite étale $k$-group scheme of order $p^m$.

**Exercise 3.** Let $k$ be a field of characteristic 0. Show that every non-trivial étale group scheme over $k$ is not unipotent.

**Exercise 4.** Let $G = \mathrm{Spec}\,(\mathcal{A})$ be a finite locally free commutative group scheme over a scheme $S$.

(1) Let $\mathcal{A}^D = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, \mathcal{O}_S)$ be the set of $\mathcal{O}_S$-linear homomorphism. Show that $\mathcal{A}^D$ with the maps $\mathcal{A}^D \otimes_{\mathcal{O}_S} \mathcal{A}^D \to \mathcal{A}^D$, $\mathcal{O}_S \to \mathcal{A}^D$, given by app lying $\mathrm{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to the comultiplication $m : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}$ and counit $\mathcal{A} \to \mathcal{O}_S$, becomes an $\mathcal{O}_S$-algebra.
(2) Notations being as in (1), show that $\mathcal{A}^D$ with the maps $\mathcal{A}^D \to \mathcal{A}^D \otimes_{\mathcal{O}_S} \mathcal{A}^D$, $\mathcal{A}^D \to \mathcal{O}_S$, $\mathcal{A}^D \to \mathcal{A}^D$, given by applying $\mathrm{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to the multiplication $m : \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$, unit $\mathcal{O}_S \to \mathcal{A}$, and coinverse $\mathcal{A} \to \mathcal{A}$, becomes an $\mathcal{O}_S$-Hopf algebra.
(3) Check that the $S$-group scheme $G^D := \mathrm{Spec}_{\mathcal{O}_S}(\mathcal{A}^D)$ is isomorphic to the group scheme $\mathrm{Hom}(G, \mathbb{G}_m)$ defined as follows:

[1] If you want your solutions to be corrected, please hand them in just before the lecture on June 8, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

Given an $S$-scheme $T$, $\mathrm{Hom}(G, \mathbb{G}_m)(T) := \mathrm{Hom}_{T-\mathrm{Grp.Sch}}(G \times_S T, \mathbb{G}_{m,T})$.

(4) Show that there is a canonical isomorphism between $(G^D)^D$ and $G$.
(5) Show that $\mu_{n,S}^D = (\mathbb{Z}/n\mathbb{Z})_S$ and vise versa.
(6) Show that if $S$ is a scheme of characteristic $p > 0$, then $\alpha_{p,S}^D = \alpha_{p,S}$.
(7) Show that if $S = \mathrm{Spec}\,(k)$, then $G$ is of multiplicative type iff $G^D$ is étale.
(8) Show that if $S = \mathrm{Spec}\,(k)$, then $G$ is unipotent iff $G^D$ is connected. (Hint: To prove that $G^D$ is connected implies that $G$ is unipotent, you can use the fact that over an algebraically closed field of characteristic $p > 0$ any finite commutative group scheme which is not of the form $\alpha_{p,k}$, $(\mathbb{Z}/l)_k$ ($l$ a prime number), 0, and $\mu_{p,k}$, has a non-trivial subgroup scheme.)