May 25, 2016

Algebraic Groups

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Exercise sheet 6[1]

Exercise 1. Let G be an affine group scheme over k. Show that the following statements are equivalent:

(1) The group scheme G is unipotent, i.e. every finite dimensional representation V of G has a non-zero fixed point;
(2) There is a directed set I such that $G = \varprojlim_{i\in I} G_i$, where G_i are unipotent algebraic groups;
(3) There is a directed set I such that $G = \varprojlim_{i\in I} G_i$, where G_i are unipotent algebraic groups and each projection $G \to G_i$ is surjective;
(4) For every quotient $G \twoheadrightarrow G'$, if G' is algebraic and affine then G' is unipotent;
(5) For any non-trivial subgroup $G' \subseteq G$ there is a non-trivial group homomorphism $G' \to \mathbb{G}_a$.

(Hint: Use the definition to show that any quotient of a unipotent group is unipotent, so (1),(2),(3),(4) are equivalent. (3)$\Rightarrow$(5). For (5) $\Rightarrow$ (4), by (5) we can take a central normal series of G (possibly infinite) using the same method we used in the class, then the image of the central normal series in the quotient would provide a central normal series in the quotient.)

Exercise 2. Show that subgroups, quotient groups of unipotent groups, and extensions by unipotent groups are all unipotent groups. (In the class we only showed this under the assumption that G is algebraic, but it holds in general.)

Exercise 3. Let K/k be a field extension. Show that a group scheme G is unipotent over k if and only if $G \times_k K$ is unipotent over K. (In the class we only showed this under the assumption that G is algebraic, but it holds in general.)

Exercise 4. (1) Compute the group $\mathrm{Hom}_{k-\text{grp.sch}}(\mathbb{G}_{m,k}, \mathbb{G}_{m,k})$;
(2) Compute the group $\mathrm{Hom}_{k-\text{grp.sch}}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$;

[1] If you want your solutions to be corrected, please hand them in just before the lecture on June 1, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

(3) Compute the group $\mathrm{Hom}_{k-\mathrm{grp.sch}}(\mathbb{G}_{a,k}, \mathbb{G}_{m,k})$;
(4) Compute the group $\mathrm{Hom}_{k-\mathrm{grp.sch}}(\mathbb{G}_{m,k}, \mathbb{G}_{a,k})$.