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Algebraic Groups

Dr. Lei Zhang

Exercise sheet 5[1]

Exercise 1. Let S be a scheme, and let M be an abelian group. Let $\mathcal{C}$ be the category of quasi-coherent $\mathcal{O}_S$-modules equipped with an action of $\mathcal{O}_S[M]$, by which we mean an $\mathcal{O}_S$-module homomorphism $\rho : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S[M]$ satisfying the usual commutative diagrams required for an action. Let $\mathcal{D}$ be the category of M-graded $\mathcal{O}_S$-coherent module. In the class we have constructed two canonical functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Show that F, G are quasi-inverse to each other.

Exercise 2. Let $n \in \mathbb{N}$, and let k be a field of characteristic $p \geq 0$. Show that $(\mathbb{Z}/n\mathbb{Z})_k$, i.e. the constant group scheme assoicated to the abstract group $\mathbb{Z}/n\mathbb{Z}$, is of multiplicative type if and only if $p \nmid n$.

Exercise 3. Let $G = \mathbb{G}_{m,k} \times_k \mathbb{G}_{m,k}$, and let V be a representation of G. Show that

$$V = \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}} V_{(s,t)}$$

where $V_{(s,t)}$ is a subrepresentation of G on which $(a, b) \in \mathbb{G}_{m,k} \times_k \mathbb{G}_{m,k}$ acts as the multiplication by $a^s b^t$.

Exercise 4. An abelian category $\mathcal{C}$ is called semisimple if any object $A \in \mathcal{C}$ is a direct sum of simple objects in $\mathcal{C}$, i.e. object which has only 0 or itself as its subobjects.

(1) Show that the category of representations of a diagonalizable group scheme is a semisimple category. What are the simple objects in this category?
(2) Show that the category of representations of group schemes of multiplicative type is a semisimple category. What are the simple objects in this category?

[1]If you want your solutions to be corrected, please hand them in just before the lecture on May 25, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.