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Algebraic Groups

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Exercise sheet 4[1]

Exercise 1. Let k be a field. Show that any affine subgroup of $\mathbb{G}_{m,k}$ is $\mu_{n,k}$ for some $n \in \mathbb{N}$.

Exercise 2. Let M be an abelian group, and let k be a field. Show that $D(M)$ is an algebraic group, i.e. a group scheme of finite type if and only if M is finitely generated.

Exercise 3. Show that there is a canonical inclusion $\mu_{n,k} \subseteq \mathbb{G}_{m,k}$ sending an n-th root of unity of $\Gamma(T, \mathcal{O}_T)$ to itself, viewed as an element in $\Gamma(T, \mathcal{O}_T)^*$. What is the quotient $\mathbb{G}_{m,k}/\mu_{n,k}$?

Exercise 4. Let M be a finitely generated abelian group. An element $x \in M$ is p-torsion iff $x^{p^n} = 1$ for some $n \in \mathbb{N}^+$, and it is prime to p-torsion iff $x^n = 1$ for some n such that $p \nmid n$. Show that the subset of M consisting of p-torsion elements (resp. prime to p-torsion elements) is a subgroup of M, and we will denote it as M_p (resp. $M_{\nmid p}$). Show that

(1) $D(M/M_{\nmid p}) = D(M)^0$;
(2) $D(M/M_p) = D(M)_{\text{red}}$, where $D(M)_{\text{red}}$ stands for the reduced subscheme of $D(M)$. In general G_{red} is not a subgroup scheme of G for any algebraic group G over k, but it is true in this case.
(3) Show that if M is a torsion group, then $D(M/M_p) \cong D(M)_{\text{ét}}$.

Exercise 5. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between two abelian categories.

(1) If F is an equivalence then F is exact.
(2) If $\mathcal{C}$ is the category of abelian groups, $\mathcal{D}$ is the category of diagonalizable group schemes over k, and F is the functor sending $M \mapsto D(M)$. Show that F is exact.
(3) If $\mathcal{D}$ is a full subcategory of another abelian category $\mathcal{D}'$, give an example to show that the inclusion $\mathcal{D} \to \mathcal{D}'$ is not always exact.

[1]If you want your solutions to be corrected, please hand them in just before the lecture on May 18, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

(4) If $\mathcal{D}$ is the category of diagonalizable group schemes and if $\mathcal{D}'$ is the category of commutative affine group schemes then $\mathcal{D} \to \mathcal{D}'$ is exact. (Hint: You can use the fact that any closed subgroup scheme of a diagonalizable group scheme is still diagonalizable, which we proved in the class.)

(5) Let's take the notations in (2), (4). Show that the functor D sending $M \mapsto D(M)$ is an exact functor from the category of abelian groups to the category of commutative affine group schemes.