May 6, 2016

Algebraic Groups

Dr. Lei Zhang

Exercise sheet 3[1]

Exercise 1. Let k be a field, and let k^s its separable closure. Let $I := \{k_i \subseteq k^s | k_i$ is a finite Galois extension of $k\}$. Then I is a directed set, i.e. a partially ordered set such that every pair of elements has an upperbound. Recall in the class we define the Galois group scheme of k as the projective limit $\mathrm{Gal}_k := \varprojlim_{k_i \in I} \mathrm{Aut}_k(k_i)$ where each $\mathrm{Aut}_k(k_i)$ is thought of as a finite constant group scheme over k and the limit is taken in the category of k-group schemes. We also denote $\mathrm{Gal}(k)$ the usual absolute Galois group of k, this is a topological group equipped with the Krull topology.

(1) Show that there is a canonical isomorphism $\mathrm{Gal}_k(k) \xrightarrow{\cong} \mathrm{Gal}(k)$, where $\mathrm{Gal}_k(k)$ is equipped with the Zariski topology.

(2) Recall that given an étale k^s-scheme X we define a *compatible Galois action* on X as an action of Gal_k on X (here X is seen as a k-scheme) such that the diagram

$$\begin{tikzcd} X \times_k \mathrm{Gal}_k \arrow[r, "\rho"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,(k^s) \times_k \mathrm{Gal}_k \arrow[r, "\phi"] & \mathrm{Spec}\,(k^s) \end{tikzcd}$$

where ρ is the action on X and ϕ is the canonical action of Gal_k on $\mathrm{Spec}\,(k^s)$. Recall also that we defined the continuous action of $\mathrm{Gal}(k)$ on X to be a group action of $\mathrm{Gal}(k)$ on $X(k^s)$ sucht that the induced map of topological spaces $X(k^s) \times \mathrm{Gal}(k) \to X(k^s)$ with discrete topology on $X(k^s)$ is continuous. Show that a continuous action of $\mathrm{Gal}(k)$ on X is the same as a compatible action of Gal_k on X.

Exercise 2. Let k be a field of characteristic $p \geq 0$, and $n \in \mathbb{N}^+$. Let $\mu_{n,k}$ be the group scheme of n-th roots of unity, i.e. as a functor it sends

[1] If you want your solutions to be corrected, please hand them in just before the lecture on May 11, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

a k-scheme T to the n-th roots of unity of $\Gamma(T, \mathcal{O}_T)$. Using the technique of Galois descent explained in the class to deduce the following fact: To give $\mu_{n,k}$ is the same as to give a group homomorphism

$$\chi : \mathrm{Gal}(k) \to \mathrm{Aut}(\mu_{n,k}(k^s)) = (\mathbb{Z}/n\mathbb{Z})^*$$

where χ sends $\sigma \in \mathrm{Gal}(k)$ to the automorphism of $\mu_{n,k}(k^s)$ sending $\zeta \mapsto \sigma(\zeta)$.

Exercise 3. (To Marco) We know that any flat locally of finite presentation morphism of schemes $f : X \to Y$ is open. Let k be a field, and let k^s its separable closure. Show that any map of rings $k^s \otimes_k k^s \to R$ is flat. Show that the diagonal morphism $\mathrm{Spec}\,(k^s) \to \mathrm{Spec}\,(k^s \otimes_k k^s)$ corresponding to $k^s \otimes_k k^s \to k^s$ sending $x \otimes y \mapsto xy$ is open if and only if k^s/k is a finite extension.

Exercise 4. Let X be a locally Noetherian scheme over a field k, and let k^s its separable closure. Show that X is connected if and only if the action of $\mathrm{Gal}(k)$ on the connected components of $X \times_k k^s$ is transitive, i.e. for any two connected components $Y_1, Y_2 \subseteq X \times_k k^s$ there exists $\sigma \in \mathrm{Gal}(k)$ such that the induced map $X \times_k k^s \xrightarrow{\mathrm{id} \times \sigma} X \times_k k^s$ sends Y_1 to Y_2.