April 29, 2016

Algebraic Groups

Dr. Lei Zhang

Exercise sheet 2[1]

Exercise 1. Let S be a scheme. Let $\mathcal{F}$ be a presheaf, i.e. a functor $(\mathrm{Sch}/S) \to (\mathrm{Sets})$. In the following we are going to define, step by step, the sheafication of $\mathcal{F}$.

(1) Let $U \in (\mathrm{Sch}/S)$. Take $\mathcal{F}^s(U)$ to be the set $\mathcal{F}(U)/\sim$, where $\sim$ is an equivalence relation defined as follows: If $a, b \in \mathcal{F}(U)$, then $a \sim b$ if and only if there is a n fppf covering $\{U_i \to U\}_{i \in I}$ such that $a|_{U_i} = b|_{U_i}$ for all $i \in I$. Show that in this way we get a presheaf $\mathcal{F}^s$ which is separated, i.e. for any fppf-covering $\{U_i \to U\}_{i \in I}$ the map $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective.

(2) Let $U \in (\mathrm{Sch}/S)$. Take $\mathcal{F}^a(U)$ to be the set of pairs $(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})$, where $\{U_i \to U\}_{i \in I}$ is a covering and $a_i \in \mathcal{F}^s(U_i)$ such that the pullback of a_i, a_j to $\mathcal{F}^s(U_i \times_U U_j)$ concide, modulo the following equivalent relation: $(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})$ is equivalent to $(\{V_j \to V\}_{j \in J}, \{b_j\}_{j \in J})$ if and only if the restriction of a_i, b_j to $\mathcal{F}^s(U_i \times_U V_j)$ coincide. Show that F^a is a sheaf.

(3) Show that the composition $\pi : \mathcal{F} \to \mathcal{F}^s \to \mathcal{F}^a$ satisfies the following universal property: Given any fppf sheaf $\mathcal{G}$ and any map $\phi : \mathcal{F} \to \mathcal{G}$ there exists a unique map $\lambda : \mathcal{F}^a \to \mathcal{G}$ such that $\phi = \lambda \circ \pi$.

(4) Show that $\mathcal{F} \to \mathcal{F}^a$ is injective if and only if $\mathcal{F}$ is separated.

Exercise 2. Let $f : G_1 \to G_2$ be map of group schemes locally of finite presentation over a scheme S.

(1) We define the kernel $\mathrm{Ker}(f)$ of f to be the S-group scheme obtained by the following pullback diagram

$$\begin{array}{ccc} \mathrm{Ker}(f) & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle e} \\ G_1 & \xrightarrow{f} & G_2 \end{array}$$

[1] If you want your solutions to be corrected, please hand them in just before the lecture on May 4, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

Show that under this definition $\mathrm{Ker}(f)$ is indeed the kernel of f, i.e. it satisfies the universal property of "kernel".

(2) Recall that in the class we have defined the cokernel $\mathrm{Coker}(f)$ of f as the fppf associated sheaf of the presheaf

$$T \mapsto \mathrm{Coker}(G_1(T) \xrightarrow{f(T)} G_2(T))$$

and if $S = \mathrm{Spec}\,(k)$ then $\mathrm{Coker}(f)$ is representable by a scheme of finite type over k. Show that when $S = k$ and the image of each $f(T)$ is normal for all k-scheme T, then $\mathrm{Coker}(f)$ is a group scheme and satisfies the universal property for a cokernel.

(3) Show that the category of commutative algebraic group over a field k is an Abelian category.

Exercise 3. Let $\alpha_{p,k} = \mathrm{Spec}\,(k[T]/T^p)$, where k is a field of characteristic p.

(1) Show that $\alpha_{p,k}$ represents the following group functor: Given a k-scheme T, we associate with T the group

$$\{x \in \Gamma(T, \mathcal{O}_T) | x^p = 0\}$$

(2) Show that the category of finite dimensional $\alpha_{p,k}$-representations is equivalent to the category of pairs (V, T), where V is a finite dimensional k-vector space and $T : V \to V$ is a linear map such that $T^p = 0$. (Hint: Use the explicit calculation we did in the class.)