July 15, 2016

Algebraic Groups

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Exercise sheet 12[1]

Exercise 1. Let k be a commutative ring. If V is a k-module, we can define see V as a functor from the category of k-algebras to the category of k-modules sending $R \mapsto V \otimes_k R$. Let GL_V be the group functor sending any k-algebra R to the group of R-linear automorphisms of $V \otimes_k R$.

(1) Show that

$$\mathrm{Lie}_{\mathrm{GL}_V}(R) = \{\mathrm{id} + A\epsilon | A : V \otimes_k R \to V \otimes_k R\} \subseteq \mathrm{GL}_V(R(\epsilon))$$

where $R(\epsilon) := R[\epsilon]/(\epsilon^2)$.

(2) Let End_V be the functor sending R to the ring of endomorphisms of $V \otimes_k R \to V \otimes_k R$. Show that there is a canonical isomorphism $\mathrm{End}_V \xrightarrow{\cong} \mathrm{Lie}_{\mathrm{GL}_V}$.

Exercise 2. Let GL_V be the group functor as in Ex.1. Show that for any k-algebra R and any $A \in \mathrm{GL}_V(R)$ the adjoint action from GL_V on $\mathrm{Lie}_{\mathrm{GL}_V}$, which we defined in the lecture, corresponds to the conjugation $M \mapsto AMA^{-1}$ ($M \in \mathrm{End}_V(R)$) via the correspondence in Ex.1.

Exercise 3. Let G be a group functor over a commutative ring k. Recall how we defined the Lie bracket on Lie_G. The adjoint action of G on its Lie-algebra Lie_G induces a map of k-group functors $G \to \mathrm{GL}_{\mathrm{Lie}_G}$. Taking the functor $\mathrm{Lie}(-)$ again, we get a map of functors

$$\mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{GL}_{\mathrm{Lie}_G}) = \mathrm{End}_{\mathrm{Lie}_G}$$

Now this map provides us a bilinear map of functors

$$\mathrm{Lie}(G) \times_k \mathrm{Lie}(G) \to \mathrm{Lie}(G)$$

and we use $[-,-]$ to denote the bilinear map. We have shown in the class that if G is linear algebraic over a field then this bilinear map satisfies all the axioms of a Lie-bracket. Now we will show that this holds indeed for any group functor over any base.

[1]If you want your solutions to be corrected, please hand them in just before the lecture on July 20, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

(1) Let $G = \mathrm{GL}_V$, the group functor defined in Ex.1. Show that for any R a k-algebra and any $A \in \mathrm{End}_V(R)$, the action of $\mathrm{id} + A\epsilon \in \mathrm{GL}_V(R(\epsilon))$ on $\mathrm{End}_V(R(\epsilon))$ is given by

$$X + Y\epsilon \mapsto (\mathrm{id} + A\epsilon)(X + Y\epsilon)(\mathrm{id} - A\epsilon)$$

where $X + Y\epsilon \in \mathrm{End}_V(R(\epsilon))$.

(2) The automorphism $\mathrm{id}+A\epsilon$ regarded as an element in $\mathrm{GL}_V(R(\epsilon))$ is mapped to $\mathrm{GL}_{\mathrm{Lie}_{\mathrm{GL}_V}}(R(\epsilon))$ via the adjoint representation, so by Ex.1, it is of the form $\mathrm{id} + T\epsilon$, where $T \in \mathrm{End}_{\mathrm{Lie}(\mathrm{GL}_V)}(R(\epsilon))$. This means that whenever there is an element

$$X + Y\epsilon \in \mathrm{Lie}_G(R(\epsilon)) = \mathrm{End}_V(R(\epsilon))$$

we have the equality

$$(\mathrm{id} + T\epsilon)(X + Y\epsilon) = X + (Y + T(X))\epsilon$$

Now show that $T(X) = AX - XA$. This means that the Lie-bracket in $\mathrm{Lie}_{\mathrm{GL}_V} = \mathrm{End}_V$ is of the form $[A, X] = AX - XA$ so it verifies all the axioms for a Lie-algebra.

(3) Mimic we we did in the class to show that for any group functor G over any base k the Lie-bracket $[-, -]$ verifies all the axioms for a Lie-algebra.