June 30, 2016

Algebraic Groups

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Exercise sheet 11[1]

Exercise 1. Let k be a perfect field. Show that G is reductive if and only if $R(G)$ is a torus.

Exercise 2. Let G be a smooth connected group scheme over k. Show that

(1) If G is semisimple, then every normal smooth connected commutative closed subgroup is trivial, and the converse is true if k is perfect.
(2) If G is reductive, then every normal smooth connected commutative closed subgroup is a torus, and the converse is true if k is perfect.

A linear algebraic group G is called linearly reductive *if every finite dimensional representation is semisimple.*

Exercise 3. Show that an affine commutative algebraic group is linearly reductive if and only if it is of multiplicative type. (Hint: We have seen that any commutative linear algerbaic group is of multiplicative type if and only if there is no non-trivial map from this group to $\mathbb{G}_m$.)

In the lecture we have seen that for any scheme X/k locally of finite type acting on by a linear algebraic group G and for any two closed subscheme $Y, Z \subseteq X$, the transporter functor *$T_G(Y, Z)$ sending T/k to $\{g \in G(T) | gY_T \subseteq Z_T\}$ is representable by a closed subscheme of G.*

Exercise 4. Let G be an affine algebraic group over a field k, and let $H \subseteq G$ be a closed subgroup scheme.

(1) Show that the normalizer $N_G(H)$ defined by $\{x | x \in G, xNx^{-1} = N\}$ is a closed subgroup scheme of G.
(2) Show that the centralizer $C_G(H)$ defined by

$$\{x | x \in G, x \text{ centralizes } H \text{ in } G\}$$

is representable by a closed subgroup scheme of G.
(3) How to define the center of G?

[1]If you want your solutions to be corrected, please hand them in just before the lecture on July 6, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.