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# Algebraic Groups

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## Exercise sheet 10[1]

**Exercise 1.** Let $k$ be a field. Let

$$\mathbb{S}^1 := \operatorname{Spec}\left(k[X,Y]/(X^2+Y^2-1)\right)$$

be the unit sphere group with group structure given by

$$(x,y)\cdot(x',y') = (xx'-yy', xy'+x'y)$$

(1) Let $\mathbb{G}_m := \operatorname{Spec}\left(k[S,T]/(ST-1)\right)$. Show that if $k=\mathbb{C}$, then the map $\mathbb{S}^1 \to \mathbb{G}_m$ given by $S \mapsto X+iY$ and $T \mapsto X-iY$ is an isomorphism of group schemes;
(2) Show that if $k=\mathbb{R}$ then the ideal $I := (X, Y-1)$ is not a principal ideal of $\mathbb{R}[X,Y]/(X^2+Y^2-1)$; (Hint: If $I$ was a principal ideal, then it is generated by a polynomial $f(X,Y)$ with coefficients in $\mathbb{R}$. But over $\mathbb{C}$ the ideal $I$ is generated by $X+iY-i$. Then $X+iY-i$ and $f(X,Y)$ differ by an invertible element in $\mathbb{C}[X,Y]/(X^2+Y^2-1)$. Show that all invertible elements in $\mathbb{C}[X,Y]/(X^2+Y^2-1)$ are of the form $a(X+iY)^n$ with $a\in k^*$ and $n\in\mathbb{N}^+$.)
(3) Show that $k[S,T]/(ST-1)$ is a principal ideal domain;
(4) Show that if $k=\mathbb{R}$ then $\mathbb{S}^1$ is not isomorphic to $\mathbb{G}_m$ as schemes (not to say as $\mathbb{R}$-schemes).
(5) Recall the analytic argument we talked about in the lecture, and make sense of the argument. Note that that argument only shows that $\mathbb{G}_{m,\mathbb{R}} \neq \mathbb{S}^1_{\mathbb{R}}$ as $\mathbb{R}$-schemes, **but does not imply that they differ as schemes**.

**Exercise 2.** Use Borel's fixed point theorem to show that over an algebraically closed field $k$, any smooth connected solvable algebraic group $G$ is trigonalizable. (Hint: In fact we have already did it in the class. Take a faithful representation $G \hookrightarrow \mathrm{GL}_{n,V}$. Then $G$ acts on the flag variety $X$ of $V$. But we know that $X$ is proper, so there is a fixed point. This means $G$ fixes a flag in $V$...)

[1]If you want your solutions to be corrected, please hand them in just before the lecture on June 29, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

*Recall that we have seen in the class that for any trigonalizable smooth connected algebraic group $G/k$ we have an exact sequence*

$$1 \to G^u \to G \to G^m \to 1$$

*of algebraic groups, where $k = \bar{k}$, $G^u$ is unipotent, and $G_m$ is a torus. And we know that the exact squence always splits. If $s$ is a section, then $s(G^m)$ is a maximal torus of $G$. Any two maximal tori are conjugate by an element in $G^u(k)$ and any maximal torus is obtained in this way.*

**Exercise 3.** Use what we have shown in the class to prove that for any smooth affine algebraic group $G$ over $k = \bar{k}$ and any two maximal tori in $G$ are conjugate by an element in $G^0(k)$.

**Exercise 4.** Use what we have shown in the class to prove that for any connected smooth affine algebraic group $G$ over $k = \bar{k}$ and any two maximal smooth connected unipotent subgroup of $G$ are conjugate by an element in $G(k)$. Show also that any such subgroup is of the form $B^u$, where $B$ is a Borel subgroup of $G$ and $B^u$ is the maximal unipotent subgroup of $B$.