

Addendum to the course on November 27, 2018

Dear students,

here an addendum to [Mil13, II, Thm.3.10]. Milne's proof contains a few typos and the main points are not really underlined, so we go through the proof of the statement.

Theorem 1. *Let G be a finite group. Let M be a G -module such that $H_T^1(H, M) = H_T^2(H, M) = 0$ for all subgroups $H \subset G$. Then $H_T^r(G, M) = 0$ for all $r \in \mathbb{Z}$.*

In the proof below, we constantly use that by definition $H_T^r(G, M) = H^r(G, M)$ for $r \geq 1$, we do not repeat this point each time.

Proof. When G is cyclic, one applies [Mil13, Prop.3.4].

Assume next G is solvable, i.e. there is a sequence of subgroups $\dots \subset G_{i+1} \subset G_i \subset \dots \subset G = G_0$ such that for all $i \in \mathbb{Z}$ we have that $G_{i+1} \neq G_i$ is normal and the quotient group G_i/G_{i+1} is abelian. In particular G_0/G_1 is abelian, and as such, has a cyclic quotient $G_0/G_1 \twoheadrightarrow C \neq 0$, which is a quotient $G_0 = G \twoheadrightarrow C$ of G by precomposing with the quotient homomorphism $G_0 \twoheadrightarrow G_0/G_1$. We set $H = \text{Ker}(G \twoheadrightarrow C)$. By [Mil13, Prop.1.34] one has an exact sequence

$$(1) \quad 0 \rightarrow H_T^r(C, M^H) \rightarrow H_T^r(G, M) \rightarrow H_T^r(H, M) \quad \forall r \geq 1$$

Thus

$$(2) \quad H_T^1(C, M^H) = H_T^2(C, M^H) = 0$$

and by [Mil13, Prop.3.4], since C is cyclic, one has

$$(3) \quad H_T^r(C, M^H) = 0 \quad \forall r \in \mathbb{Z}.$$

Since M viewed as a H -module verifies the assumption of the theorem and $|H| < |G|$, one has by induction on $|G|$ that

$$(4) \quad H_T^r(H, M) = 0 \quad \forall r \in \mathbb{Z}.$$

Thus by (1) one obtains

$$(5) \quad H_T^r(G, M) = 0 \quad \forall r \geq 1.$$

The next point is to show that $H_T^0(G, M) = 0$, which is to say that any element in $x \in M^G$ is a norm $x = \text{Nm}_G(z)$ for a certain $z \in M$. One has $x \in M^G \subset M^H$ thus x induces a class in $H_T^0(C, M^H) = (M^H)^C / \text{Nm}_C M^H = M^G / \text{Nm}_C M^H$, and this latter group is 0 by (3). Thus there is $y \in M^H$ such that $x = \text{Nm}_C(y)$. By (4) applied to $r = 0$

one has $H_T^0(H, M) = M^H / \text{Nm}_H M = 0$ thus there is $z \in M$ such that $y = \text{Nm}_H(z)$. Thus one obtains

$$(6) \quad x = \text{Nm}_C \text{Nm}_H(z) = \text{Nm}_G(z) \text{ thus } H_T^0(G, M) = 0.$$

Hence bringing (5) and (6) together reads

$$(7) \quad H_T^r(G, M) = 0 \quad \forall r \geq 0.$$

We now wish to show that $H_T^r(G, M) = 0$ for $r < 0$. To this aim we use the standard method for shifting the cohomological degree. One defines the G -module M' by the exact sequence

$$(8) \quad 0 \rightarrow M' \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} M \xrightarrow{\text{valuation}} M \rightarrow 0$$

where the evaluation map assigns to $(g \otimes m) \in \mathbb{Z}[G] \otimes_{\mathbb{Z}} M$ the element $g \cdot m \in M$. On the other hand one has an extension of Shapiro's lemma in [Mil13, Prop.3.1]

Claim 2. For G a finite group and $N = \mathbb{Z}[G] \otimes_{\mathbb{Z}} M$ an induced G -module, one has $H_T^r(G, N) = 0$ for all $r \in \mathbb{Z}$.

Granted this general form of Shapiro's lemma, we conclude from the long exact sequence in Tate cohomology associated to (8) that

$$(9) \quad H_T^r(G, M) = H_T^{r+1}(G, M') \quad \forall r \in \mathbb{Z}.$$

From (7) for $r = 0, 1$ and from (9) we obtain in particular

$$(10) \quad H_T^1(G, M') = H_T^2(G, M') = 0$$

thus we can apply (7) to M' and conclude

$$(11) \quad H_T^{-1}(G, M) = H_T^0(G, M') = 0.$$

Replacing M by M' in the preceding argument we conclude

$$(12) \quad H^{-1}(G, M') = 0,$$

but this is precisely saying

$$(13) \quad H^{-2}(G, M) = 0.$$

We keep going. This proves

$$(14) \quad H_T^r(G, M) = 0 \quad \forall r \in \mathbb{Z}$$

and proves the theorem when G is solvable.

We now address the general case. Recall that for $H \subset G$ an inclusion of finite groups and M a G -module, one has a restriction homomorphism

$$(15) \quad \text{Res} : H_T^r(G, M) \rightarrow H_T^r(H, M).$$

For $r \geq 1$ this is the easy functor of restriction in cohomology. For $r \leq -2$ this comes from the more complicated functor in homology defined by

$$(16) \quad M_G \rightarrow M_H, x \mapsto \sum s^{-1} \cdot x$$

where s goes through a system of representatives of the set G/H . See [Har11, p.25, 1.-8]. It is a good exercise to see that (16) is well defined! To go ahead, one needs the generalization of [Mil13, Cor.1.33] to Tate cohomology, which also follows from a generalization of [Mil13, Proposition 1.30]:

Claim 3 ([Har11], p.32, Ex. 4). Let $G_p \subset G$ be a p -Sylow subgroup, then the restriction of $\text{Res} : H_T^r(G, M) \rightarrow H_T^r(G_p, M)$ to the subgroup $H_T^r(G, M)$ of elements killed by multiplication by a p -power is injective.

As finite p -groups are solvable (see [MilGrp, Corollary 6.7] for a proof), applying (14) to G_p for all p we conclude that

$$(17) \quad \text{the torsion subgroup of } H_T^r(G, M) \text{ is } 0 \forall r \in \mathbb{Z}.$$

Finally we know by the corollary [Mil13, Cor.1.31] of Shapiro's lemma that for $r > 0$, $|G|H_T^r(G, M) = 0$ for any G -module M . We conclude by (9) that

$$(18) \quad |G|H_T^r(G, M) = 0 \forall r \in \mathbb{Z}.$$

Thus (17) together with (18) finish the proof. \square

REFERENCES

- [Har11] Harari, D.: *Cohomologie galoisienne et théorie des nombres*, <https://www.math.u-psud.fr/~harari/enseignement/cogal/poly.pdf>
- [Mil13] Milne, J.: *Class Field Theory*, <http://www.jmilne.org/math/CourseNotes/CFT.pdf>
- [MilGrp] Milne, J.: *Group Theory*, <http://www.jmilne.org/math/CourseNotes/GT.pdf>