October 30, 2018

Addendum to the course on October 30, 2018

Dear students,

I'm sorry for the moment of confusion. Fortunately, Floris came at the end of the lecture with an immediate repair. Let me explain.

Let G be a group. Recall

Definition 1. *A $\mathbb{Z}[G]$-module is injective I if and only if $\mathrm{Hom}_{\mathbb{Z}[G]}(-, I)$ is exact.*

It follows

Corollary 2. *Let $0 \to I' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be an exact sequence of G-modules where I' is injective. Then the induced sequence $0 \to (I')^G \to M^G \to (M'')^G \to 0$ is exact.*

Proof. We apply $\mathrm{Hom}_{\mathbb{Z}[G]}(-, I')$ to this exact sequence. Thus the sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(M'', I') \to \mathrm{Hom}_{\mathbb{Z}[G]}(M, I') \to \mathrm{Hom}_{\mathbb{Z}[G]}(I', I') \to 0$$

is exact. In particular, there is a $\theta \in \mathrm{Hom}_{\mathbb{Z}[G]}(M, I')$ which maps to the identity in $\mathrm{Hom}_{\mathbb{Z}[G]}(I', I')$, thus splits ι. Thus one has a splitting $M = I' \oplus M''$ which sends x to $\theta(x) \oplus \pi(x)$, which is G-equivariant. This in particular implies the corollary.

□

Proposition 3. *Let $0 \to M' \to M \to M'' \to 0$ be a sequence of $\mathbb{Z}[G]$-modules. Then it induces a long exact sequence of abelian groups*

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\partial_0} H^1(G, M') \to \dots$$
$$\to H^r(G, M') \to H^r(G, M) \to H^r(G, M'') \xrightarrow{\partial_r} H^{r+1}(G, M') \to \dots$$

Proof. The proof we initiated in the class works word by word knowing all the lifts we took in the injective resolution can be made G-invariants via the corollary. Let us do it. We choose injective resolutions $I'^{\bullet}, I^{\bullet}, I''^{\bullet}$ of M', M, M'' and we know we can lift the exact sequence of the Ms to an sequence of $\mathbb{Z}[G]$-modules (i.e. of abelian groups)

$$0 \to I'^{\bullet} \to I^{\bullet} \to I''^{\bullet} \to 0 \tag{1}$$

where the vertical complexes are exact except at the zeroth level, where the zeroth kernels are M', M and M'', and where, for any r, the sequence

$$0 \to I'^{r} \to I^{r} \to I''^{r} \to 0 \tag{2}$$

is exact as well. The corollary tells us that more is true: the sequences

$$0 \to (I')^G \to (I^r)^G \to (I''^r)^G \to 0 \tag{3}$$

of $\mathbb{Z}$-modules are exact for all r. Now we repeat what we did in class. *Definition of* ∂_r: let $x''^r \in (I''^r)^G$ with $d''^r(x''^r) = 0$. Then x''^r lifts to $x^r \in (I^r)^G$ thanks to (3). Then $d^r(x^r)$ lies in $(I'^{r+1})^G \subset (I^{r+1})^G$ as it maps to 0 in $(I''^{r+1})^G$ (here we do not need the corollary as we only use left exactness). And then $d'^{r+1}(d^r(x^r)) = d^{r+1}(d^r(x^r)) = 0$. So we have

$$x''^r \in (I''^r)^G, d''^r(x''^r) = 0 \mapsto d^r(x^r) \in (I'^{r+1})^G, d'^{r+1}(d^r(x^r)) = 0. \tag{4}$$

If we modify x''^r by $d''^{r-1}x''^{r-1}$, with $x''^{r-1} \in (I''^{r-1})^G$ (so we apply again (3)), we modify the lift x^r by $d^{r-1}x^{r-1}$ for some $x^{r-1} \in (I^{r-1})^G$ lifting x''^{r-1} (again by (3)), so $d^r(x^r)$ does not change. Finally another lift x^r differs by some $x'^r \in (I'^r)^G$. So $d^r(x^r)$ is modified by $d'^r x'^r$. Thus the composite

(5)

$$x''^r \in (I''^r)^G, d''^r(x''^r) = 0 \mapsto d^r(x^r) \in (I'^{r+1})^G, d'^{r+1}(d^{r+1}(x^r)) = 0$$
$$\to \mathrm{Im}(d^r(x^r)) \in H^{r+1}(G, M')$$

factors through

$$\partial_r : H^r(G, M'') \to H^{r+1}(G, M'). \tag{6}$$

Exactness at the spot $H^r(G, M'')$: let $\bar{x}''^r$ with $\partial_r(\bar{x}''^r) = 0$, where $\bar{x}''^r$ is the residue class of $x''^r \in (I''^r)^G$ in $H^r(G, M'')$. Then with the notations above $d^r(x^r) = d^r(x'^r)$ where $x'^r \in (I'^r)^G$. On the other hand, $x^r - x'^r$ is another lift of x''^r. And it verifies $d^r(x^r - x'^r) = 0$. Thus we proved $\mathrm{Ker}\partial_r \subset \mathrm{Im}(H^r(G, M) \to H^r(G, M''))$. Vice-versa, we want to show $\mathrm{Im}(H^r(G, M) \to H^r(G, M''))$ lies in $\mathrm{Ker}\partial_r$. This in turn is obvious as if x^r is a representative of $\bar{x}^r \in H^r(G, M)$ then we can take x^r as a lift of its image in $(I''^r)^G$ and by definition $d^r(x^r) = 0$.

The exactness that the other spots is simpler, we do not do it.

□