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0.1. **Topological groups.** Let $X$ be a *topological space*, and $x \in X$ be a point. Then a *neighbourhood* $N$ of $x$ is a subset of $X$ which contains an open subset $U$ of $X$ which contains $x$, so $X \supset U \ni x$. A *basis of neighbourhoods* of $x$ is a non-empty set $\mathcal{N}$ of neighbourhoods such that for any open subset $U$ containing $x$, there is a $N \in \mathcal{N}$ with $U \supset N \ni x$. One has

**Lemma 1.** *Let $(X, x, \mathcal{N})$ be as above. Then*
(a) *for all $N_1, N_2$ in $\mathcal{N}$, there is a $N' \in \mathcal{N}$ such that $N' \subset N_1 \cap N_2$.*

Recall from Milne, *Infinite Galois extensions* the

**Proposition 2** (Prop.7.2, first part)**.** *We assume in addition $X = G$ is a topological group and $x = 1$ is the unit element. Then*
(a) *Lemma 1;*
(b) *for all $N \in \mathcal{N}$, there is a $N' \in \mathcal{N}$ with $N'N' \subset N$;*
(c) *for all $N \in \mathcal{N}$, there is a $N' \in \mathcal{N}$ with $N' \subset N^{-1}$;*
(d) *for all $N \in \mathcal{N}$, for all $g \in G$, there is a $N' \in \mathcal{N}$ with $N' \subset gNg^{-1}$;*
(e) *for all $g \in G$, $\{gN, N \in \mathcal{N}\}$ is a basis of neighbourhoods of $g$.*

We have seen this and this is just stemming from the definition of a topological group.

There is a *characterization* of the topology of a topological group using (a), (b), (c), (d), and this is what we haven't discussed to the end.

**Proposition 3** (Prop.7.2, second part)**.** *Let $G$ be a group, $\mathcal{N}$ be a nonemtpy set of subsets of $G$ satisfying (a) (b) (c) (d). Then there is a unique topology on $G$ for which (e) holds.*

*Proof.* We have to reconstruct the set $\mathcal{U}$ of open subsets of $G$. We define $\mathcal{U}$ by saying that $\emptyset \in \mathcal{U}$ and else if $U \neq \emptyset$, then that $U \in \mathcal{U}$ if and only if for any $g \in U$, there is a $N \in \mathcal{N}$ such that $gN \subset U$. Indeed: $\mathcal{U}$ is not empty as it contains $G$ and $\emptyset$. If $\{U_\alpha\} \subset \mathcal{U}$ for a family indexed by $\alpha \in A$, then by definition $\cup_{\alpha \in A} U_\alpha \in \mathcal{U}$ and if $U_1, U_2 \in \mathcal{U}$, and $g \in U_1 \cap U_2$, then there are $N_1, N_2$ with $gN_i \subset U_i$ thus $gN_1 \cap gN_2 \subset U_1 \cap U_2$ and by (1), there is a $N' \in \mathcal{N}$ with $N' \subset N_1 \cap N_2$ thus $gN' \subset gN_1 \cap gN_2 \subset U_1 \cap U_2$. So $\mathcal{U}$ *is a topology* on $G$. So far, this is all in Milne's book.

*However, what is missing in the book is to show that if $G$ is a topological group, $\mathcal{N}$ is a basis of neighbourhoods of $1$, then $\mathcal{U}$ is the topology on $G$ and not another one.* We now prove this. We have to show that any $N \in \mathcal{N}$ is a neighbourhood of 1 *for the topology defined by* $\mathcal{U}$, which is equivalent to saying that for any $N \in \mathcal{N}$, there is a $U_N \in \mathcal{U}$, with $N \supset U \ni 1$. So we have to define this $U$. We set

$$U_N = \{g \in N \text{ such that there is a } N' \in \mathcal{N} \text{ with } gN' \subset N.\}.$$

Clearly $U_N \subset N$ and $1 \in U_N$ taking $N' = N$. We have to show that $U_N \in \mathcal{U}$, that is we have to show that if $g \in U_N$, then there is a $N'' \in \mathcal{N}$ with $gN'' \subset N$. To do this, by definition of $U_N$, there is a $N' \in \mathcal{N}$ with $gN' \subset N$. By (b) there is $N'' \in \mathcal{N}$ with $N''N'' \subset N'$. Thus $gN'' = gN'' \cdot 1 \subset gN''N'' \subset gN' \subset N$.

□

Now we come to the Krull topology. One has the $G \supset G(S) = \{g \in G, gs = s \ \forall s \in S\}$ defined for finite subsets $S \subset \Omega$ where $\Omega \supset F$ is a Galois extension and $G = \mathrm{Aut}(\Omega/F)$. We have seen that $\mathcal{N} := \{G(S),\ S \text{ finite} \subset \Omega\}$ fulfill (a), (b), (c), (d). One has:

**Lemma 4.** $\{gG(S),\ \text{for all } G(S) \in \mathcal{N},\ \text{all } g \in G.\}$ *is a basis of open sets for the topology of* $G$.

*Proof.* As $G(S)$ is a group, for any $g \in G(S)$, one has $gG(S) = G(S)$. □

So in conclusion: our Krull topology is really defined with a basis of *open* neighbourhoods, and $G(S) \subset G$ is a subgroup and normal if $S$ is $G$-stable (see the proof in the course).

**Remark 5.**

If you want to play, show that for $\mathbb{R}$ with the additive group structure, the standard topology endows $\mathbb{R}$ with the structure of a topological group. Then show that $\mathcal{N}$ consisting of $[a, b]$ with $a < 0 < b$, together with $\emptyset$ and $\mathbb{R}$ is a basis of neighbourhoods of 1 for the standard topology. And show that if we define $\mathcal{N}'$ as we defined $\mathcal{N}$ but with the condition $a \le 0 \le b$, then still $\mathcal{N}'$ verifies (a) (b) (c) (d), and the topology defined by $\mathcal{U}$ is the discrete topology.

0.2. **Graphs.** We have seen, for $S \subset \Omega$ finite, $G = \mathrm{Aut}(\Omega/F)$-stable, where $\Omega \supset F$ is a Galois extension, $G(S) \subset G$ is normal. We denote by $\mathcal{S}$ the set of all such $S$. We denote by $\pi_S : G \to G/G(S)$ the projection, it is a surjective continuous homomorphism to a finite group $G/G(S)$ (see the course). The product map

$$G \xrightarrow{\prod_{S \in \mathcal{S}} \pi_S} \prod_S G/G(S)$$

is injective and identifies the image with

$$\mathrm{Im}(\prod_{S\in\mathcal{S}} \pi_S) = \cap \Gamma_{S,T}$$

where the intersection is taken over all pairs $(S,T)$ in $\mathcal{S}$ with $T \subset S$, and $\Gamma_{S,T}$ is defined as follows: $T \subset S$ if and only if $G(T) \supset G(S)$. This defines the projection $\pi_{S,T} : G/G(S) \twoheadrightarrow G/G(T)$ which is a surjective homomorphism of finite discrete topological groups. Then define

$$G/G(S) \times G/G(T) \supset \gamma_{S,T} := \{(x,y), y = \pi_{S,T}(x)\},$$

and

$$\Gamma_{S,T} := \prod_{S'\neq S, S'\neq T, S'\in\mathcal{S}} G/G(S') \times \gamma_{S,T} \subset \prod_{S\in\mathcal{S}} G/G(S).$$

We want to show

**Proposition 6.**

$$\mathrm{Im}(\prod_{S\in\mathcal{S}} \pi_S) \subset (\prod_{S\in\mathcal{S}} \pi_S)(\prod_{S} G/G(S))$$

*is closed.*

Indeed if true, then as the product group is compact, so is $G$, as $\prod_{S\in\mathcal{S}} \pi_S$ is a continuous isomorphism of $G$ onto its image. Now Proposition 6 follows from $\Gamma_{S,T}$ being closed. And as the product group is endowed with the product topololgy, this is equivalent to the $\gamma_{S,T} \subset G/G(S) \times G/G(T)$ being closed.

Now forget our situation. Remember only that we showed that the Krull topology on $G$ is *Hausdorff.*

**Proposition 7.** *Let $f : X \to Y$ be a continuous map of f topological spaces, where $Y$ is Hausdorff. Then $\gamma_f = \{(x,y), y = f(x)\} \subset X \times Y$ is closed, where $X \times Y$ is endowed with the product topology.*

*Proof.* The map $(1,f) : X \times Y \to Y \times Y$ is continuous and $\gamma_f$ is the inverse image of $\gamma_{\mathrm{id}_Y} \subset Y \times Y$ where $\mathrm{id}_Y$ is the identity of $Y$. So it is enough to prove the statement for $\gamma_{\mathrm{Id}_Y}$, which is usually called *the diagonal of* $Y$. But $(a,b) \in Y \times Y \setminus \gamma_{\mathrm{Id_Y}}$ if and only if $a \neq b$, thus as $Y$ is Hausdorff, there are open subsets $U_a \ni a, U_b \ni b$ of $Y$ with $U_a \cap U_b = \emptyset$. Thus $U_a \times U_b \subset Y \times Y \setminus \gamma_{\mathrm{Id_Y}}$, and is an open for the product topology. Thus $Y \times Y \setminus \gamma_{\mathrm{Id_Y}}$ is open. ◻

We apply Proposition 7 to $f = \pi_{S,T}$, then $\gamma_f = \gamma_{S,T}$. This finishs the proof of Proposition 6.

**Remark 8.** For those who know what is the Zariski topology: it is *not* a Hausdorff topology. Yet graphs $\gamma_f \subset X \times Y$ are often *still closed* if we endow $X \times Y$ with the Zariski topology. The point is that the Zariski topology on $X \times Y$ is *not* the product topology.

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