NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 9

HÉLÈNE ESNAULT, LARS KINDLER

Exercise 1. Let k be a field of characteristic $\neq 2$ and let A be a central simple algebra over k such that $\dim_k(A) = 4$. This exercise shows that there are $a, b \in k^\times$, such that A is isomorphic to the generalized quarternion algebra $H(a, b; k)$. By Wedderburn's theorem there exists a division algebra over k, such that $A \cong M_n(D)$ for some $n \geq 1$. Thus either $A \cong M_2(k) = H(1, 1; k)$ or $A \cong D$ is a division algebra. From now on we assume that $A = D$ is a division algebra.

(a) Pick $x \in A \setminus k$ and let $k[x] \subseteq A$ be the sub-k-algebra generated by x. Show that $k[x]$ is a field and that $[k[x] : k] = 2$.
(b) Show that $C_A(k[x]) = k[x]$ where $C_A(k[x])$ is the centralizer of $k[x]$ in A.
(c) Let σ be the unique nontrivial k-automorphism of $k[x]$ and show that there exists $J \in A^\times$ such that $\sigma(y) = JyJ^{-1}$ for all $y \in k[x]$. Show that $J^2 \in k^\times$, and define $b := J^2$.
(d) Pick $I \in k[x]$ and $a \in A^\times$ such that $I^2 = a$. Prove that $A \cong H(a, b; k)$.

Exercise 2. Let D be a finite division algebra and let k denote its center (a finite field).

(a) Remark that $\dim_k(D) = n^2$ for some $n \in \mathbb{N}$.
(b) Use the Skolem-Noether theorem to show that if $L \subseteq D$ is a maximal subfield then $D^\times = \bigcup_{\alpha \in D^\times} \alpha L^\times \alpha^{-1}$ as abelian groups.
(c) Conclude that $L = D$.
(d) Conclude that the Brauer group of a finite field is trivial.

Exercise 3. Let K be a nonarchimedean local field, i.e. a complete discretely valued field with finite residue field k. We assume that $\mathrm{char}(K) = 0$ and as usual write $\mathcal{O}_K$ for the valuation ring of K, and $\mathfrak{m}_K$ for its maximal ideal.

(a) Let D be a central division algebra over K, with $[D : K] = n^2$. Prove the following statements.
  (i) The absolute value $|\cdot|$ on K extends uniquely to an absolute value on D, i.e., to a map $D \to \mathbb{R}_{\geq 0}$ such that $|x| = 0$ iff $x = 0$, and such that for all $x, y \in D$ we have $|xy| = |x||y|$ and $|x + y| \leq \max\{|x|, |y|\}$.
  (ii) If $q = \#k$, define the "valuation v_D" such that $|x| = (1/q)^{v_D(x)}$ for all $x \in D$. Define
$$\mathcal{O}_D := \{x \in D | v_D(x) \geq 0\}, \quad \mathfrak{m}_D := \{x \in D | v_D(x) > 0\}.$$
Show that $\mathcal{O}_D$ consists of the elements of D which are integral over $\mathcal{O}_K$.
  (iii) $\mathfrak{m}_D$ is a two-sided ideal in $\mathcal{O}_D$ and $\mathfrak{m}_K\mathcal{O}_D = \mathfrak{m}_D^e$ for some $0 < e \leq n$.
  (iv) $d := \mathcal{O}_D/\mathfrak{m}_D$ is a field and $f := [d : k] \leq n$.
  (v) $\mathcal{O}_D$ is a free $\mathcal{O}_K$-module of rank
$$n^2 = \dim_k(\mathcal{O}_D/\mathfrak{m}_K\mathcal{O}_D) = ef.$$

If you want your solutions to be corrected, please hand them in just before the lecture on January 3, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

(vi) Conclude that $e = f = n$.

(vii) Write $d = \mathcal{O}_D/\mathfrak{m}_D\mathcal{O}_D = k[a]$ and let $\alpha \in D$ be a lift of a. Then $K[\alpha]$ is a maximal subfield of D and splits D. Show that $K[\alpha]/K$ is unramified.

(b) If D/K is a central division algebra, and $L \subseteq D$ a maximal subfield unramified over K, then L/K is Galois with Galois group $\mathrm{Gal}(d/k)$. Let $\sigma \in \mathrm{Gal}(L/K)$ be the lift of the Frobenius automorphism of d/k. Show that there exists $\alpha \in D$ such that $\sigma(x) = \alpha x \alpha^{-1}$ for all $x \in L$. Show that $v_D(\alpha) \mod \mathbb{Z}$ is independent of the choice of α.

(c) Show that the above construction gives a well-defined map $\mathrm{inv}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$, $[D] \mapsto \alpha \mod \mathbb{Z}$. We will prove next year that inv_K is in fact an isomorphism.